Algebraic and Geometric Multiplicities

Notation: Let A be an $m \times m$ matrix and let $\lambda \in \mathbb{C}$. Then the algebraic multiplicity and geometric multiplicity of λ in A are respectively,

$m_A(\lambda) = \mathrm{mult}_\lambda(ch_A)$,

$\nu_A(\lambda) = \dim_{\mathbb{C}} E_A(\lambda) = m - \mathrm{rank}(A - \lambda I)$.

Example: If $A = J(\lambda, m)$ is a Jordan block, then $m_A(\lambda) = m$ and $\nu_A(\lambda) = 1$.

Theorem 1: (Sum Formula) If $A = \mathrm{Diag}(B, C)$, then

$$\begin{aligned} m_A(\lambda) &= m_B(\lambda) + m_C(\lambda), \\ \nu_A(\lambda) &= \nu_B(\lambda) + \nu_C(\lambda). \end{aligned}$$

Theorem 2: In fact, we have

$$\begin{aligned} \mathrm{ch}_A(\lambda) &= \mathrm{ch}_B(\lambda) \cdot \mathrm{ch}_C(\lambda), \\ E_A(\lambda) &= E_B(\lambda) \oplus E_C(\lambda). \end{aligned}$$

Corollary: If $J = \mathrm{Diag}(J_{11}, J_{12}, \ldots, J_{ij}, \ldots)$ is a Jordan matrix with Jordan blocks $J_{ij} = J(\lambda_i, k_{ij})$, then

$$\begin{aligned} \nu_J(\lambda_i) &= \#(\text{Jordan blocks } J_{ij} \text{ with e.value } \lambda_i), \\ m_J(\lambda_i) &= \text{sum of the sizes } k_{ij} \text{ of the Jordan blocks} \\ &\quad\ \text{with eigenvalue } \lambda_i. \end{aligned}$$