The Jordan Canonical Form – Examples

Example 1: Given $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$, find its JCF and P.

Here: $\left.\begin{array}{l} \mathrm{ch}_A(t) = (t+1)^2 \\ \nu_A(-1) = 1 \end{array}\right\} \Rightarrow J = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$.

We want to find an invertible matrix P such that

(1) $\quad P^{-1}AP = J$ or, equivalently, $AP = PJ$.

Write $P = (\vec{v}_1|\vec{v}_2)$. Then, since $AP = (A\vec{v}_1|A\vec{v}_2)$ and $PJ = (-\vec{v}_1|\vec{v}_1 - \vec{v}_2)$, we see that equation (1) is equivalent to:

1) $A\vec{v}_1 = -\vec{v}_1$ or $(A+I)\vec{v}_1 = \vec{0}$
2) $A\vec{v}_2 = \vec{v}_1 - \vec{v}_2$ or $(A+I)\vec{v}_2 = \vec{v}_1$
3) $\vec{v}_1, \vec{v}_2$ are lin. indep. ($\Leftrightarrow P$ is invertible)

Observations: a) Equations 1) and 2) imply

4) $(A+I)^2\vec{v}_2 = \vec{0}$.

b) If we pick a vector $\vec{v}_2 \in \mathbb{C}^2$ which satisfies 4) and define $\vec{v}_1$ by equation 2), then $\vec{v}_1, \vec{v}_2$ satisfy 1) and 2).

However: we have to pick $\vec{v}_2$ carefully such that condition 3) is also satisfied.

c) It turns out: it is enough to require that

$$\vec{v}_1 := (A+I)\vec{v}_2 \neq \vec{0}, \text{ i.e. } \vec{v}_2 \notin E_A(-1) = \{c(1,-1)^t\}.$$

Take $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so $P = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$.

Example 2: If $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & 0 \\ 1 & -2 & 3 \end{pmatrix}$, find its JCF and P.

Here: $\left.\begin{array}{l} \mathrm{ch}_A(t) = (t-2)^3 \\ \nu_A(2) = 2 \end{array}\right\} \Rightarrow J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$

Find an invertible matrix $P = (\vec{v}_1|\vec{v}_2|\vec{v}_3)$ such that $AP = PJ$. Since $AP = (A\vec{v}_1|A\vec{v}_2|A\vec{v}_3)$ and $PJ = (2\vec{v}_1|\vec{v}_1 + 2\vec{v}_2|2\vec{v}_3)$, this is equivalent to:

1) $A\vec{v}_1 = 2\vec{v}_1$ or $(A - 2I)\vec{v}_1 = \vec{0}$
2) $A\vec{v}_2 = \vec{v}_1 + 2\vec{v}_2$ or $(A - 2I)\vec{v}_2 = \vec{v}_1$
3) $A\vec{v}_3 = 2\vec{v}_3$ or $(A - 2I)\vec{v}_3 = \vec{0}$
4) $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are lin. indep. ($\Leftrightarrow P$ is invertible)

Again: Equations 1) and 2) imply: 5) $(A - 2I)^2\vec{v}_2 = \vec{0}$.

Thus: pick a vector $\vec{v}_2 \in E_A^2(2)$ not in $E_A(2)$;

define $\vec{v}_1 = (A - 2I)\vec{v}_2$;

pick $\vec{v}_3 \in E_A(2)$, linearly independent from $\vec{v}_1, \vec{v}_2$.

Here: $E_A(2) = \{c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\}$, $E_A^2(2) = \mathbb{C}^3$.

Take $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$; also take $\vec{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

Then: $P = (\vec{v}_1|\vec{v}_2|\vec{v}_3) = \begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.