Generalized Eigenvectors

Definition: Let A be an $m \times m$ matrix and $\lambda \in \mathbb{C}$. A vector $\vec{v} \in \mathbb{C}^m$ is called a *generalized λ-eigenvector* of A *of order* $\leq p$ if

$$(A - \lambda I)^p \vec{v} = \vec{0}.$$

If, in addition, $(A - \lambda I)^{p-1}\vec{v} \neq \vec{0}$, then we say that $\vec{v}$ has (precise) order (or degree) p. The space

$$\begin{aligned} E_A^p(\lambda) &= \text{Nullsp}((A - \lambda I)^p) \\ &= \{\vec{v} \in \mathbb{C}^m : (A - \lambda I)^p \vec{v} = \vec{0}\}. \end{aligned}$$

is called the *p-th generalized λ-eigenspace* of A, and its dimension

$$\nu_A^p(\lambda) := \dim E_A^p(\lambda) = m - \text{rank}((A - \lambda I)^p)$$

is the *p-th generalized geometric multiplicity* of A with respect to λ.

Remarks: 1) For $p = 1$ we recover the usual eigenspace:

$$E_A^1(\lambda) = E_A(\lambda), \text{ and so } \nu_A^1(\lambda) = \nu_A(\lambda).$$

2) It is clear that $E_A^p(\lambda) \subset E_A^{p+1}(\lambda), \forall p$. Thus

$$\{0\} \subset E_A^1(\lambda) = E_A(\lambda) \subset E_A^2(\lambda) \subset \ldots E_A^p(\lambda) \subset \ldots$$

Example: If $A = J(\lambda, m)$ is a Jordan block, then

$$E^p_{J(\lambda,m)} = \mathrm{Nullsp}((A - \lambda I)^p) = \langle \vec{e}_1, \ldots, \vec{e}_p \rangle,$$

if $p \le m$, and $E^p_{J(\lambda,m)} = \mathbb{C}^m$, if $p \ge m$. Thus

$$\nu^p_{J(\lambda,m)}(\lambda) \;=\; \begin{cases} p & \text{if } p \le m, \\ m & \text{if } p \ge m. \end{cases}$$

Theorem 5 (Properties of gen. eigenvectors):

(a) If $A = \mathrm{Diag}(B, C)$ and if $p \ge 1$, then

$$E^p_A(\lambda) \;=\; E^p_A(\lambda) \oplus E^p_B(\lambda);$$

in particular, $\nu^p_A(\lambda) = \nu^p_B(\lambda) + \nu^p_C(\lambda)$.

(b) If $A = PBP^{-1}$, then

$$E^q_A(\lambda) \;=\; PE^q_B(\lambda), \quad \text{for all } q \ge 1;$$

in particular, $\nu^q_A(\lambda) = \nu^q_B(\lambda)$, for all $q \ge 1$.

(c) If J is a Jordan canonical form of A, then

$$\nu^p_A(\lambda) - \nu^{p-1}_A(\lambda) = \#(\text{Jordan blocks } J(\lambda, k_{ij}) \text{ of JCF } J \text{ of size } k_{ij} \ge p).$$

(d) We have that

$$\nu^p_A(\lambda) = \nu^{p+1}_A(\lambda) \Rightarrow \nu^p_A(\lambda) = \nu^{p+q}_A(\lambda), \text{ for all } q \ge 1.$$

Corollary: The numbers $\nu^p_A(\lambda_i)$ determine the JCF of A (by taking second differences).

Example 1: Let A be a matrix with characteristic polynomial $\mathrm{ch}_A(t) = (t-7)^5$ and generalized geometric multiplicities

$$\begin{aligned}\nu_A^*(7) &= (\nu_A^1(7), \nu_A^2(7), \nu_A^3(7), \ldots) \\ &= (2, 4, 5, 5, 5, \ldots)\end{aligned}$$

Find the associated Jordan canonical form J of A.

Solution: By using Theorem 5 (or its Corollary) we obtain:

$$\begin{array}{lllll}
\nu_A^4 - \nu_A^3 = 5-5 = 0 & \Rightarrow & 0 \text{ blocks of size } \geq 4 & & \\
 & & & \} & 1 \text{ bl. size } 3 \\
\nu_A^3 - \nu_A^2 = 5-4 = 1 & \Rightarrow & 1 \text{ block of size } \geq 3 & & \\
 & & & \} & 1 \text{ bl. size } 2 \\
\nu_A^2 - \nu_A^1 = 4-2 = 2 & \Rightarrow & 2 \text{ blocks of size } \geq 2 & & \\
 & & & \} & 0 \text{ bl. size } 1 \\
\nu_A^1 - \nu_A^0 = 2-0 = 2 & \Rightarrow & 2 \text{ blocks of size } \geq 1 & &
\end{array}$$

and hence J has:
0 blocks of size 1×1
1 block of size 2×2
1 block of size 3×3
0 blocks of size 4×4 etc.

Thus: $J = \mathrm{Diag}(J(7,3), J(7,2)) = \left(\begin{array}{ccc|cc} 7 & 1 & 0 & 0 & 0 \\ 0 & 7 & 1 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ \hline 0 & 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 0 & 7 \end{array}\right)$

(up to order of the blocks).

Example 2: Find the Jordan canonical form of

$$A = \begin{pmatrix} 7 & 0 & 0 & 1 & 0 \\ 0 & 7 & 0 & 0 & 0 \\ 1 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 1 & 0 & 0 & 7 \end{pmatrix}.$$

Solution: Step 1: Find the characteristic polynomial.

Expanding $\det(A - tI)$ along the first row yields

$$\mathrm{ch}_A(t) = (-1)^5 \det(A - tI) = (t-7)^5.$$

Thus, $m_A(7) = 5$. Moreover, since

$$B = A - 7I = (\vec{e}_3 | \vec{e}_5 | \vec{0} | \vec{e}_1 | \vec{0}),$$

we have $\nu_A(7) = 5 - \mathrm{rk}(B) = 2 = \#(\text{Jordan blocks})$.

Step 2: Calculate the generalized geometric multiplicities.

Since $B(\vec{v}_1|\vec{v}_2|\vec{v}_3|\vec{v}_4|\vec{v}_5) = (B\vec{v}_1|B\vec{v}_2|B\vec{v}_3|B\vec{v}_4|B\vec{v}_5)$ and since $B\vec{e}_i = i$-th column of B, we see that

$$B^2 = (B\vec{e}_3|B\vec{e}_5|B\vec{0}|B\vec{e}_1|B\vec{0}) = (\vec{0}|\vec{0}|\vec{0}|\vec{e}_3|\vec{0}),$$

and similarly

$$B^3 = B \cdot B^2 = (B\vec{0}|B\vec{0}|B\vec{0}|B\vec{e}_3|B\vec{0}) = (\vec{0}|\vec{0}|\vec{0}|\vec{0}|\vec{0}),$$

and so B, B^2, and B^3 have ranks $3, 1,$ and 0 respectively. Moreover, clearly $B^p = 0$, for all $p \geq 3$.

Thus, since $\nu_A^p(7) = 5 - \mathrm{rank}(B^p)$, we see that the generalized geometric multiplicities are

$$\nu_A^*(7) = (2, 4, 5, 5, \ldots).$$

Step 3: Find the Jordan blocks by the method of second differences.

Since $\mathrm{ch}_A(t) = (t-7)^5$ and since the generalized geometric multiplicities are

$$\nu_A^*(7) = (2, 4, 5, 5, \ldots),$$

we can conclude by Example 1 that the Jordan canonical form J of A is

$$J = \mathrm{diag}(J(7,3), J(7,2)) = \left(\begin{array}{ccc|cc} 7 & 1 & 0 & 0 & 0 \\ 0 & 7 & 1 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ \hline 0 & 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 0 & 7 \end{array}\right).$$