Powers of Complex Numbers

Definition: If $c_n = a_n + ib_n$ is a sequence of complex numbers such that the limits

$$a = \lim_{n\to\infty} a_n \quad \text{and} \quad b = \lim_{n\to\infty} b_n$$

exist, then we say that $c := a + ib$ is the *limit* of c_n and write

$$\lim_{n\to\infty} c_n = a + ib.$$

Theorem 2: If $\alpha \in \mathbb{C}$ then

a) If $|\alpha| < 1$ then $\lim_{n\to\infty} \alpha^n = 0$.

b) If $|\alpha| > 1$ then $\lim_{n\to\infty} \alpha^n = \infty$.

c) If $|\alpha| = 1$ then $\lim_{n\to\infty} \alpha^n$ exists if and only if $\alpha = 1$.

Corollary: $\lim_{n\to\infty} \alpha^n$ exists if and only if $\alpha = 1$ or $|\alpha| < 1$.

Note: a) If $|\alpha| < 1$ then the powers α^n spiral in to 0.

b) If $|\alpha| > 1$ then the powers α^n spiral out to ∞.

c) If $|\alpha| = 1$ and $\alpha \neq 1$ then the powers α^n run around on the unit circle.

Example: Powers of the numbers

$\alpha = \frac{5}{12}(2+i)$, $\beta = \frac{11}{24}(2+i)$ and $\gamma = \cos(\frac{2\pi}{100}) + i\sin(\frac{2\pi}{100})$: