Geometric Series of Matrices

Definition: Let T be any square matrix. Then the sequence $\{S_n\}_{n\geq 0}$ defined by

$$S_n = I + T + \ldots + T^{n-1}, \quad S_0 = I,$$

is called the geometric series generated by T. The series converges if the sequence $\{S_n\}_{n\geq 0}$ converges; we then write

$$\sum_{n=0}^{\infty} T^n = \lim_{n\to\infty} S_n.$$

Theorem 11: The geometric series generated by T converges if and only if

(1) $\quad |\lambda_i| < 1,$ for each eigenvalue λ_i of T.

If this condition holds, then $I - T$ is invertible and we have

(2) $$S_n := \sum_{k=0}^{n-1} T^k = (I-T)^{-1}(I-T^n),$$

and hence the series converges to

(3) $$\sum_{k=0}^{\infty} T^k = (I-T)^{-1}.$$

Remark: Note that condition (1) is equivalent to:

$$\lim_{n\to\infty} T^n = 0. \tag{4}$$