Stochastic Matrices and Markov Chains

Definition: A real $p \times q$ matrix $A = (a_{ij})$ is called stochastic if

(1) $$a_{ij} \geq 0, \text{ for all } i, j;$$

(2) $$\Sigma_p \cdot A = \Sigma_q,$$

where $\Sigma_p = \underbrace{(1, 1, \ldots, 1)}_{p} \in \mathbb{R}^p$.

A (homogeneous) Markov chain is a discrete linear system

$$\vec{v}_{n+1} = A\vec{v}_n$$

in which $\vec{v}_n$ is a stochastic vector, and

A is a stochastic matrix.

Remarks: 1) Condition (2) means: the sum of each column of A is equal to 1.

2) Markov chains were named after A.A. Markov (1856 – 1922) by A.N. Kolmogorov in 1935.

Properties: 1) A, B stochastic $\Rightarrow$ AB stochastic.

2) If A is a square stochastic matrix, then

a) $\lambda_1 = 1$ is a regular eigenvalue of A;

b) $|\lambda_i| \leq 1$ for every eigenvalue λ_i of A.