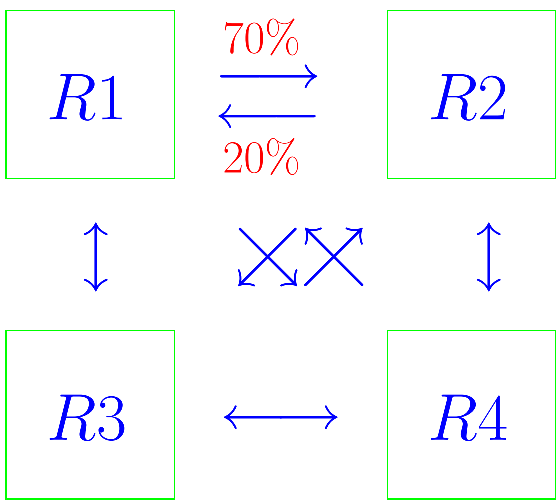
Shipping of Commodities

Situation: Four regions (or cities) $R1$, $R2$, $R3$, $R4$ ship (non-renewable) commodities (such as antique paintings, rental cars) among themselves according to the following diagram:



The figures represent the percentage of goods each region ships per week. (Percentage: in terms of goods present)

All unmarked arrows: 10%

Today's distribution:

$R1$ has r_1 of all the goods
$R2$ has r_2 of all the goods
$R3$ has r_3 of all the goods
$R4$ has r_4 of all the goods

Question: What happens in the long run?

Note: This is a Markov chain $\vec{v}_{n+1} = A\vec{v}_n$ with

$$A = \frac{1}{10}\begin{pmatrix} 1 & 2 & 1 & 1 \\ 7 & 6 & 1 & 1 \\ 1 & 1 & 7 & 1 \\ 1 & 1 & 1 & 7 \end{pmatrix} \quad \text{and} \quad \vec{v}_0 = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix}.$$

Qualitative Solution: Perron's theorem $\Rightarrow$ there is a vector $\vec{p} = (p_1, p_2, p_3, p_4)^t$ such that

– $\vec{p}$ represents the distribution of goods among the regions in the long run:

$$\lim_{n\to\infty} \vec{v}_n = \lim_{n\to\infty} A^n \vec{v}_0 = \vec{p}$$

– $\vec{p}$ does not depend on the initial distribution $\vec{v}_0$

– $p_j > 0$, for $j = 1, \ldots, 4$, i.e. no region is completely drained in the long run.

Quantitative Solution: Find the Perron vector $\vec{p}$:

By row reduction (or by inspection)

$E_A(1) = Solsp(A - I) = \{c(6, 16, 11, 11)^t : c \in \mathbb{C}\}$,

and hence the Perron vector $\vec{p}$ is given by

$$\vec{p} = \frac{1}{\Sigma_4 \vec{x}} \vec{x} = \frac{1}{44}(6, 16, 11, 11)^t = (.14, .36, .25, .25)^t$$

where $\vec{x} = (6, 16, 11, 11)^t$. Thus:

$R1$ has 14% of all the goods in the long run
$R2$ has 36% of all the goods in the long run
$R3$ has 25% of all the goods in the long run
$R4$ has 25% of all the goods in the long run