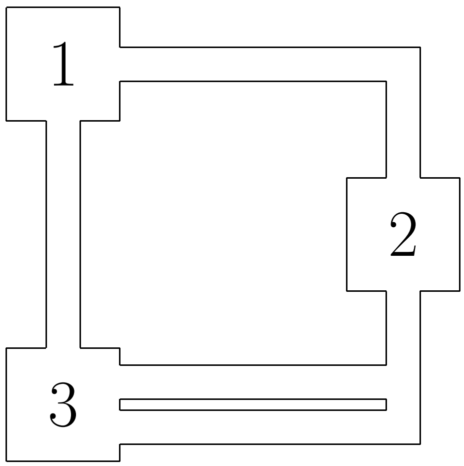
A Rat Maze II

Consider the following system of three chambers connected by passages as shown.



At time $t = 0$, a rat is placed in one of the chambers (say in chamber 1).

Each minute thereafter, the rat is driven out of its present chamber by some stimulus and is prevented from re-entering immediately.

Assume: the rat chooses the exits of each chamber at random.

Question: What is the probability that the rat is in a certain chamber in the long run?

Note: If s_i denotes the state that the rat is in chamber i, then we have a Markov chain with transition matrix

$$A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

Qualitative Solution: A is primitive because

$$A^2 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 5 & 2 & 4 \\ 4 & 7 & 2 \\ 3 & 3 & 6 \end{pmatrix}.$$

Thus, by Perron's Theorem there is a vector $\vec{p} = (p_1, p_2, p_3)^t$ such that:

– each p_j represents the probability that the rat is in chamber j in the long run:

$$\lim_{n\to\infty} \vec{v}_n = \lim_{n\to\infty} A^n \vec{v}_0 = \vec{p}$$

– $\vec{p}$ does not depend on the initial distribution $\vec{v}_0$, i.e. on where the rat was originally placed;

– $p_j > 0$, for $j = 1, \ldots, 3$, i.e. no chamber is excluded in the long run.

Quantitative Solution: Find the Perron vector $\vec{p}$:

$$\vec{p} = \frac{1}{27}(8, 10, 9)^t \doteq (.296, .370, .333)^t$$

Thus, in the long run:

The rat spends 29.6% of its time in chamber 1,
37.0% of its time in chamber 2,
33.3% of its time in chamber 3.