Math 211

Course Overview

Chapter 1: The Integers

– Divisibility, greatest common divisor

– The extended Euclidean Algorithm

– The GCD-criterion and applications, Euclid's Lemma

– Application: Solving linear Diophantine equations without/with constraints

– Prime numbers and the Unique Factorization Theorem. Applications: irrationality, GCD-formula

Chapter 2: Modular Arithmetic

– The calculus of remainders: definition and properties of $a \equiv b \pmod{m}$, the power-mod algorithm, the cancellation law

– Solving the congruence equation $ax \equiv b \pmod{m}$

– Modular arithmetic and the ring $\mathbb{Z}/m\mathbb{Z}$; special case: the field $\mathbb{F}_p$

– The Chinese Remainder Theorem

– Fermat's Little Theorem and applications

Chapter 3: Polynomials

– Complex numbers: complex conjugates, absolute values, geometrical representation, polar form, multiplication rule, De Moivre's formula, n-th roots, Euler's formula

– Polynomials: coefficients, degree, the ring $R[x]$, properties of the degree function

– Divisibility and the division algorithm. Applications: the Remainder Theorem and the Factor Theorem; the substitution method for finding $\text{rem}(f, g)$

– The greatest common divisor (of 2 polynomials), the extended Euclidean algorithm; the GCD-criterion, Euclid's Lemma (for polynomials)

– Irreducible polynomials: basic properties, connection with roots. The Quadratic Formula (and discriminants)

– The Unique Factorization Theorem for $F[x]$. Applications: the GCD-formula, the multiplicity of a root

– Factoring methods: (a) over $\mathbb{Q}$: the Rational Root Test, Gauss's Lemma, the modular test; (b) over $\mathbb{C}$: the Fundamental Theorem of Algebra; (c) over $\mathbb{R}$: the Fundamental Theorem of Algebra for $\mathbb{R}[x]$

Chapter 4: Interpolation, Approximation and the Geometry of $\mathbb{R}^n$

– The Lagrange Interpolation Polynomial: (i) the formula method; (ii) the linear algebra method; the Remainder Formula

– The Least Square Method via orthogonal projection onto a subspace of $\mathbb{R}^n$

– The geometry of $\mathbb{R}^n$: (a) points and vectors: dot product, length, distance, angles; Law of Cosines, Rule of Pythagoras, Cauchy-Schwarz inequality, triangle inequality. (b) equations of lines, planes, linear sets and of subspaces of $\mathbb{R}^n$.

– A distance problem: the orthogonal projection of a vector onto a subspace (matrix method). Application to the Least Square Method

– Orthogonal and orthonormal bases of a subspace. The orthogonal projection formula (via an orthogonal basis). The Gram-Schmidt method for finding an orthogonal basis. Orthogonal matrices.

Chapter 5: Matrix Polynomials and Discrete Linear Systems

– Evaluating Matrix Polynomials I: (i) Diagonable case: The Diagonalization Theorem: eigenvalues, eigenvectors; the characteristic polynomial $\mathrm{ch}_A(t)$; determinants

– Evaluating Matrix Polynomials I: (ii) Non-diagonable case: Jordan blocks $J(\lambda, k)$, Jordan matrices and Jordan's Theorem

– Evaluating Matrix Polynomials II: The Cayley-Hamilton Theorem. The Generalized Remainder Theorem (substitution method, formula method)

– Evaluating Matrix Polynomials III: The Spectral Decomposition Theorem. Constituent matrices and how to compute them

– Applications: discrete linear systems, difference equations (example: Fibonacci numbers, golden ratio), companion matrix; Markov chains

Chapter 6: The Jordan Canonical Form

– Algebraic and geometric multiplicities of eigenvalues (Sum Formula, Invariance Property) and their relation to the Jordan canonical form J of a matrix A

– Finding P such that $P^{-1}AJ = J$ (for $m \leq 3$)

– Properties of generalized eigenvectors and generalized geometric multiplicities. Application: how to determine J via the method of 2nd differences

Chapter 7: Powers of Matrices

– The limit of a sequence of complex numbers; powers of a number

– Rules for limits of matrices, power convergent matrices. Eigenvalue criterion for a matrix to be power convergent

– Finding $\lim_{n\to\infty} A^n$. Further properties of constituent matrices

– Properties of stochastic matrices and their eigenvalues; primitive stochastic matrices, Perron's theorem, Perron vector; applications to Markov chains.