Let A be a square matrix. Then we have proved the following theorems:

Theorem 7.7 (Page 320) If $|\lambda_i| < 1$ for every eigenvalue of A, then A is power convergent and $\lim_{n\to\infty} A^n = 0$.

Recall that an eigenvalue λ of A is regular if the Jordan Canonical form of A has no Jordan blocks of size greater than 1. Equivalently the multiplicity of λ as a root of $\mathrm{cp}_A(t)$ (the algebraic multiplicity of λ) is equal to $\dim E_A(\lambda)$ (the geometric multiplicity of λ).

Theorem 7.8 (Page 320) If $\lambda_1 = 1$ is a regular, dominant eigenvalue of A, then A is power convergent and we have

$$\lim_{n\to\infty} A^n = E_{10} \neq 0,$$

where E_{10} is the first constituent matrix of A associated to $\lambda_1 = 1$. Moreover, the other constituent matrices E_{1k} associated to $\lambda_1 = 1$ are equal to zero:

$$E_{11} = \cdots = E_{1,m_1-1} = 0.$$

We have observed that in the case of the rat-maze problem, which has transition matrix

$$A = \begin{pmatrix} 0 & 1/2 & 1/3 \\ 1/2 & 0 & 2/3 \\ 1/2 & 1/2 & 0 \end{pmatrix},$$

the columns of $\lim_{n\to\infty} A^n = E_{10}$ are equal. (In fact, they are equal to the unique stochastic eigenvector of A for eigenvalue $\lambda = 1$). This is explained by the following theorem. Recall that an eigenvalue is simple if its algebraic multiplicity is 1.

Theorem 7.10 (page 324) If A is power convergent and 1 is a simple eigenvalue of A, then

$$\lim_{n\to\infty} A^n = E_{10} = \frac{1}{\underbrace{\vec{u}^t\vec{v}}_{\text{scalar}}} \underbrace{\vec{u}\vec{v}^t}_{\text{matrix}},$$

where: $\vec{u} \in E_A(1)$ is any non-zero 1-eigenvector of A, and
$\vec{v} \in E_{A^t}(1)$ is any non-zero 1-eigenvector of A^t.

Proof. In the present discussion it is essential to remember that $\vec{u}, \vec{v}, \vec{x}$ and $\vec{y}$ are column vectors, with $\vec{u}^t, \vec{v}^t, \vec{x}^t, \vec{y}^t$ being the corresponding row vectors, so that the matrix product $\vec{u}^t\vec{v}$ is just the usual dot product of $\vec{u}$ and $\vec{v}$, whereas the matrix product $\vec{u}\vec{v}^t$ is a square matrix of the same size as A. Let P be such that $J = P^{-1}AP$ is a Jordan matrix, which we may take to be of the form $J = \text{Diag}(J(1,1), \dots)$. Then, since $|\lambda_i| < 1$ for $i > 1$, we see that

$$(1) \quad \lim_{n\to\infty} A^n = P \lim_{n\to\infty} J^n P^{-1} = P \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} P^{-1} = P\vec{e}_1\vec{e}_1^{\,t}P^{-1},$$

where $\vec{e}_1 = (1, 0, \dots, 0)^t$. Now since $\vec{e}_1$ is a 1-eigenvector of J, it follows that $\vec{x} := P\vec{e}_1$ (= the first column of P) is a 1-eigenvector of A (because $A\vec{x} = AP\vec{e}_1 = PJ\vec{e}_1 = P\vec{e}_1 = \vec{x}$).

Similarly, $\vec{y}^t := \vec{e}_1^{\,t}P^{-1}$ (= the first row of P^{-1}) is (the transpose of) a 1-eigenvector of A^t. Indeed, since $\vec{e}_1^{\,t}J = \vec{e}_1^{\,t}$, we see that $\vec{y}^tA = \vec{e}_1^{\,t}P^{-1}A = \vec{e}_1^{\,t}JP^{-1} = \vec{e}_1^{\,t}P^{-1} = \vec{y}^t$. Taking the transpose yields $A^t\vec{y} = \vec{y}$, which means that $\vec{y}$ is a 1-eigenvector of A^t. Thus by the above equation (1) we obtain

$$\lim_{n\to\infty} A^n = P\vec{e}_1\vec{e}_1^t P^{-1} = \vec{x}\vec{y}^t = \frac{1}{\vec{x}^t\vec{y}}\vec{x}\vec{y}^t,$$

the latter because $\vec{x}^t\vec{y} = \vec{y}^t\vec{x} = \vec{e}_1^t P^{-1}P\vec{e}_i = \vec{e}_1^t\vec{e}_1 = 1$. From the construction of P, $\vec{x}$ is an arbitrary non-zero eigenvector of A with eigenvalue 1, so we may set $\vec{u} = \vec{x}$. The vector $\vec{y}$ depends on the choice of $\vec{x}$ so we cannot take $\vec{y}$ to be an arbitrary non-zero eigenvector of A^t. However 1 is a simple eigenvalue of both A and A^t so every non-zero 1-eigenvector of A^t is a non-zero multiple of $\vec{y}$. Thus if $\vec{v}$ is an arbitrary non-zero 1-eigenvector of A^t we have $\vec{y} = c\vec{v}$ for some constant $c \neq 0$. We now have

$$\lim_{n\to\infty} A^n = \frac{1}{\vec{x}^t\vec{y}}\vec{x}\vec{y}^t = \frac{1}{\vec{u}^t c\vec{v}}\vec{u}c\vec{v}^t = \frac{1}{\vec{u}^t\vec{v}}\vec{u}\vec{v}^t$$

which is Theorem 7.10.

For the rat-maze matrix

$$A = \begin{pmatrix} 0 & 1/2 & 1/3 \\ 1/2 & 0 & 2/3 \\ 1/2 & 1/2 & 0 \end{pmatrix},$$

we have observed that $\vec{u} = (8/27, 10/27, 9/27)^t$ is the unique stochastic 1-eigenvector of A. Furthermore $\vec{v} = (1,1,1)^t$ is a 1-eigenvector of A^t. Since $\vec{u}$ is stochastic $\vec{u}\cdot\vec{v} = \vec{u}^t\vec{v} = 1$ so Theorem 7.10 yields

$$\lim_{n\to\infty} A^n = \vec{u}\vec{v}^t = (\vec{u}|\vec{u}|\vec{u}) = \begin{pmatrix} 8/27 & 8/27 & 8/27 \\ 10/27 & 10/27 & /10/27 \\ 9/27 & 9/27 & 9/27 \end{pmatrix}$$