# Math 211

## October Midterm - Solutions

[3] 1. The Euclidean algorithm gives

$$938 = 1 \cdot 847 + 91, \quad 847 = 9 \cdot 91 + 28, \quad 91 = 3 \cdot 28 + 7, \quad 28 = 4 \cdot 7,$$

and so we conclude that $\gcd(938, 847) = 7$. Thus, since $847 \div 7 = 121$ and $938 \div 7 = 134$, we obtain that $\frac{847}{938} = \frac{121}{134}$, which is in lowest terms (by Theorem 1(c)).

[1] 2. (a) The term "$a$ divides $b$" means that there is an integer $k$ such that $b = ak$.

[1] The hypothesis $a|b$ means that there is an integer $k \in \mathbb{Z}$ such that $b = ak$. Similarly, since $b|c$, there is an integer $n \in \mathbb{Z}$ such that $c = bn$. Thus, $c = bn = (ak)n = a(kn)$. Since $kn \in \mathbb{Z}$, this means that $a|c$, as desired.

[1] (b) The greatest common divisor of $m$ and $n$ is the largest integer $d$ such that $d|m$ and $d|n$.

[1] (c) Euclid's Lemma states that if $a, b, c$ are non-zero integers such that $a|bc$ and $\gcd(a, b) = 1$, then $a|c$.

*Alternate version:* If $\gcd(a, b) = 1$, then $a|c$ and $b|c$ if and only if $ab|c$.

[6] 3. (a) By the Euclidean algorithm we see that $g = \gcd(123, 174) = 3$, for

$$174 = 1 \cdot 123 + 51, \quad 123 = 2 \cdot 51 + 21, \quad 51 = 2 \cdot 21 + 9, \quad 21 = 2 \cdot 9 + 3 \quad \text{and} \quad 9 = 3 \cdot 3 + 0.$$

Thus, since $3|1200$, it follows from the GCD-criterion that integer solutions exist. To find these, use back-substitution:

$$\begin{aligned} 3 &= 21 - 2 \cdot 9 = 21 - 2(51 - 2 \cdot 21) = 5 \cdot 21 - 2 \cdot 51 = 5(123 - 2 \cdot 51) - 2 \cdot 51 \\ &= 5 \cdot 123 - 12 \cdot 51 = 5 \cdot 123 - 12(174 - 123) = 17 \cdot 123 - 12 \cdot 174. \end{aligned}$$

Thus, $x_0 = 17$ and $y_0 = -12$ satisfy the equation $123x_0 + 174y_0 = g$, and hence the general solution of the given Diophantine equation is

$$\left.\begin{aligned} x &= \tfrac{12000}{3}(17) + \tfrac{174}{3}t &&= 68000 + 58t \\ y &= \tfrac{12000}{3}(-12) - \tfrac{123}{3}t &&= -48000 - 41t \end{aligned}\right\} \text{ where } t \in \mathbb{Z}.$$

[Check: $123(68000 + 58t) + 174(-48000 - 41t) = 8,364,000 + 7134t - 8,352,000 - 7134t = 12,000$.]

[4] (b) We have that

$$\begin{aligned} x \ge 0 &\iff 68000 + 58t \ge 0 &&\iff t \ge -\tfrac{68000}{58} \doteq -1172.41 &&\iff t \ge -1172, \\ y \ge 0 &\iff -48000 - 41t \ge 0 &&\iff t \le -\tfrac{48000}{41} \doteq -1170.73 &&\iff t \le -1171. \end{aligned}$$

Thus, $t$ is an integer with $-1172 \le t \le -1171$, i.e. $t = -1172$ or $t = -1171$. Substituting these values for $t$ in the above formula yields the corresponding values of $x$ and $y$. For $t = -1172$ we get $x = 68000 + 58(-1172) = 24$ and $y = -48000 - 41(1172) = 52$, and similarly for $= -1171$ we obtain $(x, y) = (82, 11)$. Thus, there are precisely two non-negative integer solutions of this equation: $(x, y) = (24, 52)$ and $(x, y) = (82, 11)$.

[3] 4. Since $x \ge 0, y \ge 0$, we have $5z \le 3x + 4y + 5z = 8$, so $z \le \frac{8}{5}$. Thus $z = 0$ or $1$. For $z = 1$ we have $3x + 4y = 8 - 5 = 3$, so $y \le \frac{3}{4}$, i.e. $y = 0$ and $x = 1$. Thus, here we have the solution $(x, y, z) = (1, 0, 1)$.

If $z = 0$, then $3x + 4y = 8$, and so $y \le 2$. For $y = 2$ we get the solution $(x, y, z) = (0, 2, 0)$ whereas $y = 1$ or $y = 0$ do not yield integer solutions. Thus, $(x, y, z) = (1, 0, 1), (0, 2, 0)$ are the only non-negative solutions.

**Comments on the submitted solutions.** 1) In Q2(a), many were unable to give the correct definition. To say that "$a$ divides $b$ means that $a$ is a divisor (or a factor) of $b$" is not a definition, but only alternate terminology. Similarly, many were unable to prove the stated property. Giving an example to verify the property is not a proof. In Q1(c), it was very sad to see that most people did not know what "Euclid's Lemma" was about.

2) In Q3(a), several forgot to write "$t \in \mathbb{Z}$" when stating the general solution. Also, some had trouble with signs.

3) In Q4, many did not follow the instruction: "without finding the general solution". Please read the question carefully.

**Test statistics:** Average: 15.4/20 = 77.2% (56 students). 2 perfect, 4 failures.