Math 211

Assignment 1 - Solutions

[3] 1. (a) The first version (table method) of the Euclidean algorithm yields:

144	144	144	144	144	144	60	60	36	12	12
804	660	516	372	228	84	84	24	24	24	12

The second version of the Euclidean algorithm (using the division algorithm) gives

$$\begin{aligned} 804 &= 5 \cdot 144 + 84 \\ 144 &= 1 \cdot 84 + 60 \\ 84 &= 1 \cdot 60 + 24 \end{aligned} \qquad \begin{aligned} 60 &= 2 \cdot 24 + 12 \\ 24 &= 2 \cdot 12 \end{aligned}$$

Thus, by both methods we see that $\gcd(144, 804) = 12$. Note that the first version required 10 steps (subtractions), whereas the second version needed only 5 steps (divisions). Thus, the second method is better since there are fewer steps.

Since $144 \div 12 = 12$ and $804 \div 12 = 67$, we obtain

$$\frac{144}{804} = \frac{12}{67},$$

which is in lowest terms.

[1] (b) The second versions of the Euclidean algorithm yields

$$\begin{aligned} 2235 &= 1 \cdot 1245 + 990 \\ 1245 &= 1 \cdot 990 + 255 \\ 990 &= 3 \cdot 255 + 225 \end{aligned} \qquad \begin{aligned} 255 &= 1 \cdot 225 + 30 \\ 225 &= 7 \cdot 30 + 15 \\ 30 &= 2 \cdot 15 \end{aligned}$$

Thus, $\gcd(1245, 2235) = 15$. Since $1245 \div 15 = 83$ and $2235 \div 15 = 149$, we see that $\frac{1245}{2235} = \frac{83}{149}$, which is in lowest terms.

[2] 2. (a) The Euclidean algorithm gives:

$$\begin{aligned} 4891 &= 1 \cdot 2701 + 2190 \\ 2701 &= 1 \cdot 2190 + 511 \\ 2190 &= 4 \cdot 511 + 146 \end{aligned} \qquad \begin{aligned} 511 &= 3 \cdot 146 + 73 \\ 146 &= 2 \cdot 73 \end{aligned}$$

Thus, $\gcd(2701, 4891) = 73$.

[2] (b) Here the Euclidean algorithm yields:

$$\begin{aligned} 32564 &= 2 \cdot 12557 + 7540 \\ 12557 &= 1 \cdot 7540 + 5017 \\ 7540 &= 1 \cdot 5017 + 2523 \end{aligned} \qquad \begin{aligned} 5017 &= 1 \cdot 2523 + 2494 \\ 2523 &= 1 \cdot 2494 + 29 \\ 2494 &= 86 \cdot 29 \end{aligned}$$

Thus, $\gcd(12557, 32854) = 29$.

[2] 3. By the division algorithm we have $1000 = 76 \cdot 13 + 12$. Thus, dividing both sides by 13 we obtain $\frac{1000}{13} = 76 + \frac{12}{13}$, and so $\frac{1000}{13} = 76\frac{12}{13}$.

[2] 4. Since $2000 = 117 \cdot 17 + 11$, there are 117 numbers between 1 and 2000 which are divisible by 17. Of these, 35 are less than 600 (because $600 = 35 \cdot 17 + 5$), and so there are $82 = 117 - 35$ numbers between 2000 and 600 which are multiples of 17.

[2] 5. Let $P(n)$ be the statement: "$n^3 - n$ is divisible by 6." Then $P(1)$ is true since $6|0 = 1^3 - 1$. Thus, assume $P(k)$ is true for some $k \geq 1$, i.e. that $6|(k^3 - k)$; we want to show that $P(k+1)$ is also true. Now $(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k+1) = k^3 + 3k^2 + 2k = (k^3 - k) + 3(k^2 + k)$. We observe that $k^2 + k = k(k+1)$ is always even (because one of k and $k+1$ is always even), so we can write $k^2 + k = 2h$ with $h \in \mathbb{Z}$. (Alternately, we could use Example 1.7 of the text see that $k^2 + k = (k^2 - k) + 2k$ is always even.) Thus, we have $(k+1)^3 - (k+1) = k^3 - k + 6h$. By the induction hypothesis, $6|(k^3 - k)$, and hence $6|(k^3 - k + 6h) = ((k+1)^3 - (k+1))$, which means that $P(k+1)$ is also true. Thus, $P(n)$ is true for every positive integer $n \geq 0$.

[2] 6. (See the MAPLE solution on the course web site for more details.)

The MAPLE command "*igcd*(12345, 54321);" returns the value 3, so $\gcd(12345, 54321) = 3$. Similarly, $\gcd(213141516171, 262524232221) = 3$.

7. (See the course web site for the Maple output).

[1] (a) The list L of elements gcd(k, 24) is defined by the command `L:= [seq(igcd(k^2,24), k=1..12)];` and Maple prints `L:= [1, 4, 3, 8, 1, 12, 1, 8, 3, 4, 1, 24]`.

The 10th element of this list is given by the command `L[10];` and Maple returns the value 4.

[1] (b) The list LL of pairs (k, gcd(k,12)) is given by the command `LL:= [seq([k,igcd(k^2,12)], k=1..12)];` Maple computes
`LL:= [[1,1],[2,4],[3,3],[4,4],[5,1],[6,12],[7,1],[8,4],[9,3],[10,4],[11,1],[12,12]]`.

The commands `LL[9];` and `LL[9,2];` yield `[9,3]` and `3`, respectively. Thus, the 9th element of LL is the list [9,3], and the 2nd element of this list is 3.

[2] (c) The function f is defined by `f:= x -> x^3 + x + 1;` and the command `f(20);` computes the value of f at $x = 20$. The answer is $f(20) = 20^3 + 20 + 1 = 8021$.

Note: Remember to put your name (in Maple text) at the beginning of your Maple homework and to add comments to explain your Maple output, as was done above.