Math 211

Assignment 2 - Solutions

[2] 1. (a) Using the Euclidean algorithm we obtain:

$$\begin{array}{rcl|rcl} 7927 &=& 1\cdot 7919+8 & 8 &=& 1\cdot 7+1\\ 7919 &=& 989\cdot 8+7 & 7 &=& 7\cdot 1 \end{array}$$

Thus, the greatest common divisor is $\gcd(7919, 7927) = 1$.

[2] (b) By back-substitution we obtain $1 = 8-7 = 8-(7919-989\cdot 8) = 990\cdot 8-7919 = 990(7927-7919)-7919 = 990\cdot 7927 - 991\cdot 7919$. Thus, $x = -991$ and $y = 990$ is a solution of $7919x + 7927y = 1$.

[1] (c) Multiplying the solution of (b) by 10 shows that $x = -9910$ and $y = 9900$ is the desired solution.

[1] 2. (a) Clearly $2|294$ and $2|816$, but 2 does not divide -9. Thus, $2|\gcd(294, 816)$ but $2 \nmid (-9)$ and hence the GCD-criterion shows that the equation cannot have any integral solutions.

[2] (b) The Euclidean algorithm yields $1628 = 2\cdot 777 + 74$, $777 = 10\cdot 74 + 37$, and $74 = 2\cdot 37$, which means that $\gcd(777, -1628) = \gcd(1628, 777) = 37$. Since $37|37$, we know by the GCD-criterion that the equation has an integer solution. By back-substitution we obtain $37 = 777 - 10\cdot 74 = 777 - 10(1628 - 2\cdot 777) = 21\cdot 777 - 10\cdot 1628$. Thus, $x = 21$ and $y = 10$ is a solution of the equation $777x - 1628y = 37$.

[2] (c) Here the Euclidean algorithm gives

$$\begin{array}{rcl|rcl} 2115 &=& 1\cdot 1233+882 & 351 &=& 1\cdot 180+171\\ 1233 &=& 1\cdot 882+351 & 180 &=& 1\cdot 171+9\\ 882 &=& 2\cdot 351+180 & 171 &=& 19\cdot 9 \end{array}$$

Thus, $\gcd(2115, 1233) = 9$. Since, $9|18$, we know by the GCD-criterion that the equation has an integral solution. By back-substitution we obtain $9 = 180-171 = 180-(351-180) = 2\cdot 180-351 = 2(882-2\cdot 351) = 2\cdot 882-5\cdot 351 = 2\cdot 882-5(1233-882) = 7\cdot 882-5\cdot 1233 = 7(2115-1233)-5\cdot 1233 = 7\cdot 2115-12\cdot 1233$. Thus, $x_0 = -12$, $y_0 = 7$ is a solution of the equation $1233x_0 + 2115y_0 = 9$, and so $x = -24$, $y = 14$ is a solution of the equation $1233x + 2115y = 18$.

[4] 3. (a) We shall follow the steps of the method for solving $mx + ny = c$ given in class:

0) The Euclidean algorithm (applied to 8249 and 8023) yields:

$$8249 = 1\cdot 8023 + 226, \quad 8023 = 35\cdot 226 + 113, \quad 226 = 2\cdot 113.$$

1) Thus, $\gcd(-8249, 8023) = \gcd(8249, 8023) = 113$. Since $113|1243$, we know that a solution exists.

2) The method of back-substitution gives $113 = 8023 - 35\cdot 226 = 8023 - 35(8249 - 8023) = 36\cdot 8023 + (-35)8249$. Thus, $x_0 = 36$, $y_0 = 35$ is a solution of the equation $8023x_0 - 8249y_0 = 113$.

3) By the formula in class (Theorem 1.5), the general solution of the original equation is given by

$$\left.\begin{array}{rcccl} x &=& \frac{1243}{113}(36) + \frac{-8249}{113}t &=& 396 - 73t\\ y &=& \frac{1243}{113}(35) - \frac{8023}{113}t &=& 385 - 71t \end{array}\right\} \text{ where } t \in \mathbb{Z}.$$

[1] (b) Here, $\gcd(1079, 1411) = 83$. But 83 does not divide $243 = 2\cdot 83 + 77$, so there are no integer solutions (x, y) of this equation.

[1] (c) Since $\gcd(123456, 654321) = 3$ and 3 does not divide 7, the equation has no integer solutions.

[1] 4. We have

$$\begin{array}{rcl} x^2 + y^2 &=& (2m+1)^2 + (2m(m+1))^2 \;=\; 4m^2 + 4m + 1 + (2m(m+1))^2\\ &=& (2m(m+1))^2 + 2(2m(m+1)) + 1 \;=\; (2m(m+1)+1)^2 \;=\; z^2, \end{array}$$

and so (x, y, z) is a Pythagorean triple. For $m = 1, 2, 3, 4$ we obtain $(x, y, z) = (3, 4, 5), (5, 12, 13), (7, 24, 25)$ and $(9, 40, 41)$, respectively.

[3] 5. See the MAPLE solution on the course Web site (mast.queensu.ca/~math211).