Math 211

Assignment 3 - Solutions

[3] 1. Let x denote the number of children and y the number of adults that attended. Then we have $375x+900y = 16500$. The Euclidean algorithm yields

$$\begin{aligned} 900 &= 2\cdot 375 + 150 \\ 375 &= 2\cdot 150 + 75 \\ 150 &= 2\cdot 75 \end{aligned}$$

so $\gcd(900, 375) = 75$. Since $75|16500$, we see that integer solutions exist. By back-substitution we get $75 = 375 - 2(900 - 2\cdot 375) = 5\cdot 375 - 2\cdot 900$, so $x_0 = 5$ and $y_0 = -2$ satisfy $375x_0 + 900y_0 = 75$. Thus, the general solution is:

$$\left.\begin{aligned} x &= \tfrac{16500}{75}(5) + \tfrac{900}{75}t &&= 1100 + 12t \\ y &= \tfrac{16500}{75}(-2) - \tfrac{375}{75}t &&= -440 - 5t \end{aligned}\right\} \text{ where } t \in \mathbb{Z}.$$

We now analyze the constaints. Since $x \ge 0$ we have $1100 + 12t \ge 0$, or $t \ge -\frac{1100}{12} \doteq -91.6667$, so $t \ge -91$. Moreover, since also $y > x$ we have $-440 - 5t > 1100 + 12t$ or $t < -\frac{1540}{17} \doteq -90.58823529$, so $t \le -91$. Thus, $t = -91$ and hence $x = 1100 + 12(-91) = 8$. (Similarly, $y = 15$.) Thus, there were 8 children at the theatre.

[4] 2. Let x denote the cent portion on the original cheque, and y the dollar portion. Then the total amount (in cents) on the original cheque is $100y + x$, and the amount paid out was $100x + y$. We thus obtain $100x + y - 350 = 2(100y + x)$, or equivalently,

$$98x - 199y = 350. \tag{1}$$

Note that x and y also satisfy the constraints

$$0 \le x \le 99, \qquad 0 \le y (\le 99). \tag{2}$$

The Euclidean algorithm gives

$$199 = 2\cdot 98 + 3, \quad 98 = 32\cdot 3 + 2, \quad 3 = 1\cdot 2 + 1.$$

Thus, gcd(199,98) = 1 and $1 = 3-2 = 3-(98-32\cdot 3) = 33\cdot 3 - 98 = 33(199-2\cdot 98)-98 = (-67)\cdot 98+33\cdot 199 = (-67)\cdot 98 + (-33)\cdot(-199)$. Therefore, the general solution of (1) is

$$\left.\begin{aligned} x &= 350(-67) + (-199)t &&= -23450 - 199t \\ y &= 350(-33) - 98t &&= -11550 - 98t \end{aligned}\right\} \text{where } t \in \mathbb{Z}.$$

We now analyze the constraints (2):

$$\begin{aligned} x \ge 0 &\iff t \le \tfrac{-350\cdot 67}{199} = -117.84 &&\iff t \le -118; \\ x \le 99 &\iff t \ge \tfrac{-350\cdot 67 - 99}{199} = -118.33 &&\iff t \ge -118. \end{aligned}$$

Thus, $t = -118$ and hence $x = 350(-67) - 199(-118) = 32$, and $y = 350(-33) - 98(-118) = 14$. Therefore, the cheque was worth \$14.32.

[3] 3. (a) Write the equation as $4x + 6y = 20 + 9z$. Since $\gcd(4, 6) = 2$, we see by the GCD-criterion that this has a solution if and only if $2|20 + 9z$, i.e. if and only if $20 + 9z = 2w$, for some $w \in \mathbb{Z}$. Now since $\gcd(2, 9) = 1$ and $2(-4) - 9(-1) = 1$, the general solution of the auxiliary equation $2w - 9z = 20$ is given by

$$w = \tfrac{20}{1}(-4) + \tfrac{-9}{1}t = -80 - 9t, \quad z = \tfrac{20}{1}(-1) - \tfrac{2}{1}t = -20 - 2t, \quad \text{for } t \in \mathbb{Z}.$$

Substituting this value for z in the original equation yields $4x+6y = 20+9(-20-2t) = -160-18t$. Now since $\gcd(4,6) = 2$ and $4(-1)+6(1) = 2$, we see that the general solution of $4x+6y = -160-18t$ is given by

$$x = \tfrac{-160-18t}{2}(-1) + \tfrac{6}{2}s = 80+9t+3s, \quad y = \tfrac{-160-18t}{2}(1) - \tfrac{4}{2}s = -80-9t-2s, \quad \text{where } s,t \in \mathbb{Z}.$$

Thus, the general solution of the original equation is

$$x = 80+9t+3s, \quad y = -80-9t-2s, \quad z = -20-2t, \quad \text{where } s,t \in \mathbb{Z}.$$

Check: $4(80+9t+3s)+6(-80-9t-2s)-9(-20-2t) = 320+36t+12s-480-54t-12s+180+18t = 20.$

[1] (b) This equation has no integral solutions. Indeed, since 3 divides 3,9 and 12, we see (by property (D3)) that $3|3x+9y+12z$, for all integers x,y,z. However, 3 does not divide 20, so there are no integers x,y,z such that $3x+9y+12z = 20$.

[4] 4. Let x denote the value of a six-pence coin, y the value of a ten-centime coin and z the value of a drachma coin (all in cents). Then we have the equation

$$35x+55y+77z = 586, \tag{3}$$

with constraints $x \geq 0, y \geq 0$ and $z \geq 0$.

To solve this equation, write it as:

$$35x+55y = 586-77z. \tag{4}$$

Since $\gcd(35,55) = 5$, this has a solution if and only if $5|(586-77z)$ or, equivalently, if

$$77z+5w = 586, \text{ for some } w \in \mathbb{Z}. \tag{5}$$

The Euclidean algorithm gives $77 = 15\cdot 5+2$, $5 = 2\cdot 2+1$; thus, $\gcd(77,5) = 1$ and, moreover, $1 = 5-2\cdot 2 = 5-2(77-15\cdot 5) = (-2)\cdot 77+31\cdot 5$. Thus, the general solution of (5) is:

$$z = 586(-2)+5t, \; w = 586\cdot 31-77t, \text{ where } t \in \mathbb{Z}.$$

Substituting for z is (4) yields $35x+55y = 155\cdot 586-77\cdot 5t$ or

$$7x+11y = 31\cdot 586-77t. \tag{6}$$

Since $\gcd(7,11) = 1$ and $7(-3)+11(2) = 1$, the general solution of (6) is

$$x = (31\cdot 586-77t)(-3)+11s, \; y = (31\cdot 586-77t)(2)-7s,$$

and hence the general solution solution of (3) is

$$\left.\begin{array}{rcl} x &=& -3\cdot 31\cdot 586+3\cdot 77t+11s, \quad z = -2\cdot 586+5t \\ y &=& 2\cdot 31\cdot 586-2\cdot 77t-7s, \end{array}\right\} \text{where } s,t \in \mathbb{Z}.$$

(Other expressions are also possible. Note that $3\cdot 31\cdot 586 = 54498$, $3\cdot 77 = 231$, $2\cdot 586 = 1172$, $2\cdot 31\cdot 586 = 36332$ and $2\cdot 77 = 154$, but we don't need this.)

We now analyse the above constraints. We have:

$$\begin{array}{llll} z \geq 0 & \iff t \geq (2\cdot 586)/5 = 234.6 & \iff & t \geq 235 \text{ (since } t \in \mathbb{Z}). \\ y \geq 0 & \iff s \leq -22t+(2\cdot 31\cdot 586)/7 = -22t+5190\tfrac{2}{7} & \iff & s \leq -22t+5190, \\ x \geq 0 & \iff s \geq -21t+(3\cdot 31\cdot 586)/11 = -21t+4954\tfrac{4}{11} & \iff & s \geq -21t+4955. \end{array}$$

We thus have the inequalitites

$$(7) \qquad -22t + 5190 \;\geq\; s \;\geq\; -21t + 4955 \quad \text{and} \quad t \;\geq\; 235.$$

From these inequalities we obtain $-22t + 5190 \geq -21t + 4955$ or $t \leq 5190 - 4955 = 235$, which, together the above inequality on t yields $t = 235$. Substituting this into the inequalities (7) gives

$$20 \;=\; -22(235) + 5190 \;\geq\; s \;\geq\; -21(235) + 4955 \;=\; 20,$$

and hence $s = 20$. Substituting $s = 20, t = 235$ into the above formulae gives $x = -93(586) + 3 \cdot 77 \cdot 235 + 11 \cdot 20 = 7$, $y = 62(586) - 154 \cdot 235 - 7 \cdot 20 = 2$, $z = (-2)586 + 5(235) = 3$, and so the value of six-pence coin was 7¢, that of the ten-centime coin was 2¢, and that of the drachma was 3¢.

Alternate method (for analyzing constraints): The above inequalities (7) describe the interior region of a triangle in the (t, s)-plane with vertices $(\frac{18166}{77}, 0) \doteq (235.9, 0)$, $(\frac{1172}{5}, \frac{1172}{35}) \doteq (234.4, 33.49)$, $(\frac{1172}{5}, \frac{1758}{55}) \doteq (234.4, 31.96)$. Thus, any integral point in this region must have $t = 235$. The line $t = 235$ meets the triangle at $(235, \frac{213}{11}) \doteq (235, 19.4)$ and at $(235, \frac{142}{7}) \doteq (235, 20.3)$. Thus we see that $(t, s) = (235, 20)$ is the only point in the triangle and so we obtain the solution $x = 7$, $y = 2$ and $z = 3$ as before.

[2] 5. Rather than writing out all the numbers < 120, it is enough to write down 2 and the odd numbers < 120, and cross out successively multiples of $3, 5, 7$. (Note that $11 > \sqrt{120}$, so we do not need to cross out any other multiples.) We then obtain the following table (Sieve of Erastothenes):

2	3	5	7	~~9~~	11	13	~~15~~	17	19
~~21~~	23	~~25~~	~~27~~	29	31	~~33~~	~~35~~	37	~~39~~
41	43	~~45~~	47	~~49~~	~~51~~	53	~~55~~	~~57~~	59
61	~~63~~	~~65~~	67	~~69~~	71	73	~~75~~	~~77~~	79
~~81~~	83	~~85~~	~~87~~	89	~~91~~	~~93~~	~~95~~	97	~~99~~
101	103	~~105~~	107	109	~~111~~	113	~~115~~	~~117~~	~~119~~.

Thus, the primes less than 120 are:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113$$

and the pairs of twin primes in that range are:

$$(3, 5),\ (5, 7),\ (11, 13),\ (17, 19),\ (29, 31),\ (41, 43),\ (59, 61),\ (71, 73),\ (101, 103),\ (107, 109).$$

[1] 6. (a) $1728 = 12 \cdot 144 = 12^3 = 2^6 3^3$;

(b) $684600 = 200 \cdot 3423 = (8 \cdot 5^2)(3 \cdot 7 \cdot 163) = 2^3 3 \cdot 5^2 \cdot 7 \cdot 163$.

[2] 7. Let $P(n)$ be the statement: $1^2 + 2^2 + \ldots + n^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$. This statement is clearly true for $n = 1$ because $\frac{1}{3}1^3 + \frac{1}{2}2^2 + \frac{1}{6}1 = 1 = 1^2$. Thus, assume that $P(n)$ is true for $n = k$, i.e. asume that $1^2 + 2^2 + \ldots + k^2 = \frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k$ (induction hypothesis). To show that $P(n)$ is also true for $n = k + 1$, note that the induction hypothesis implies that

$$\begin{aligned}
1^2 + 2^2 + \ldots + k^2 + (k+1)^2 &= (1^2 + 2^2 + \ldots + k^2) + (k+1)^2 \\
&= (\frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k) + (k+1)^2 \\
&= \frac{1}{3}k^3 + \frac{3}{2}k^2 + \frac{13}{6}k + 1 \\
&= (\frac{1}{3}k^3 + k^2 + k + \frac{1}{3}) + (\frac{1}{2}k^2 + k + \frac{1}{2}) + \frac{1}{6}k + \frac{1}{6} \\
&= \frac{1}{3}(k+1)^3 + \frac{1}{2}(k+1)^2 + \frac{1}{6}(k+1);
\end{aligned}$$

here we've used in the last step the identities $(k+1)^3 = k^3 + 3k^2 + 3k + 1$ and $(k+1)^2 = k^2 + 2k + 1$. This, therefore, shows that $P(k+1)$ is true, and so, by the principle of mathematical induction, we have proved that $P(n)$ is true for all $n \geq 1$.