Problem 1:

(a) It can be indeed observed that

\[ \bar{w}(x + y) = (\text{# of 1's in } x) + (\text{# of 1's in } y) - 2(\text{# of places where both } x \text{ and } y \text{ have a 1}). \]

Thus

\[ \bar{w}(x + y) = \bar{w}(x) + \bar{w}(y) - 2\bar{w}(xy). \]

A formal proof goes as follows. Letting \( I \triangleq \{1, 2, \ldots, n\} \), define the sets \( A \) and \( B \) as

\[ A \triangleq \{i \in I : x_i = 1\}, \]

and

\[ B \triangleq \{i \in I : y_i = 1\}. \]

Then \( |A| = \bar{w}(x) \) and \( |B| = \bar{w}(y) \). Furthermore,

\[ A \cap B = \{i \in I : x_i = 1 \text{ and } y_i = 1\}, \]

\[ A \setminus B = \{i \in I : x_i = 1 \text{ and } y_i = 0\}, \]

\[ B \setminus A = \{i \in I : y_i = 1 \text{ and } x_i = 0\}, \]

and

\[ (A \setminus B) \cup (B \setminus A) = \{i \in I : x_i + y_i = 1\}. \]

Thus

\[ |A \cap B| = \bar{w}(xy), \]

and

\[ |(A \setminus B) \cup (B \setminus A)| = |A \setminus B| + |B \setminus A| = \bar{w}(x + y). \]

Now, observe that \( A \) and \( B \) can be written as the following disjoint unions:

\[ A = (A \setminus B) \cup (A \cap B), \]

\[ B = (B \setminus A) \cup (A \cap B). \]
and

\[ B = (B \setminus A) \cup (A \cap B). \]

Hence

\[
|A| = |A \setminus B| + |A \cap B| \iff \bar{w}(x) = |A \setminus B| + \bar{w}(xy) \tag{1}
\]

\[
|B| = |B \setminus A| + |A \cap B| \iff \bar{w}(y) = |B \setminus A| + \bar{w}(xy). \tag{2}
\]

Adding equations (1) and (2) yields

\[
\bar{w}(x) + \bar{w}(y) = |A \setminus B| + |B \setminus A| + 2\bar{w}(xy),
\]

or

\[
\bar{w}(x) + \bar{w}(y) = \bar{w}(x + y) + 2\bar{w}(xy).
\]

Therefore

\[
\bar{w}(x + y) = \bar{w}(x) + \bar{w}(y) - 2\bar{w}(xy).
\]

\( \square \)

(b) From (a) we have

\[
\bar{w}(x) + \bar{w}(y) = \bar{w}(x + y) + 2\bar{w}(xy).
\]

But \(\bar{w}(xy) \geq 0\), thus

\[ \bar{w}(x) + \bar{w}(y) \geq \bar{w}(x + y). \]

\( \square \)

We have equality iff \(\bar{w}(xy) = 0\). This is equivalent to \(|A \cap B| = 0\); that is \(A \cap B = \emptyset\). Therefore to show that a necessary and sufficient condition for equality is that \(y_i = 0\) whenever \(x_i = 1\), we need to prove that

\[ A \cap B = \emptyset \iff x_i = 1 \implies y_i = 0. \]

If \(A \cap B = \emptyset\), then clearly \(x_i = 1 \implies y_i = 0\). Conversely, if we have that \(x_i = 1 \implies y_i = 0\), then \(i \in A \cap B\) is impossible (since we cannot have \(x_i = y_i = 1\)); so \(A \cap B = \emptyset\).

\( \square \)

(d) We have:

\[
d(x, z) &= \bar{w}(x + z) \\
        &= \bar{w}(x + y + y + z) \quad \text{(since } y + y = 0^n \text{ in } B^n\text{)} \\
        &= \bar{w}((x + y) + (y + z))
\]
\[ \leq \bar{w}(x + y) + \bar{w}(y + z) \quad \text{(by (b))} \]
\[ = d(x, y) + d(y, z). \]

(c) Given \(x\) and \(y\), let \(u = x + y\). Then the triangle inequality of (b) states that
\[ \bar{w}(y + u) \leq \bar{w}(y) + \bar{w}(u). \]
But \(y + u = y + (x + y) = x\); thus \(\bar{w}(y + u) = \bar{w}(x)\). Therefore, the above expression becomes
\[ \bar{w}(x) \leq \bar{w}(y) + \bar{w}(x + y), \]
or
\[ \bar{w}(x + y) \geq \bar{w}(x) - \bar{w}(y). \]

From part (b), we know that we have equality iff \(y_i = 1 \implies u_i = 0\); that is \(y_i = 1 \implies x_i + y_i = 0\); that is \(y_i = 1 \implies x_i = 1.\)

Problem 2:

First note that \(d_{\text{min}} = 3\) for \(C\); so the code can correct one error per codeword. The distance computations \((d(w, c), c \in C)\) for the nearest neighbor decoding rule yield the following:

<table>
<thead>
<tr>
<th>codewords</th>
<th>(a) (w = 0110101)</th>
<th>(b) (w = 0101110)</th>
<th>(c) (w = 1011001)</th>
<th>(d) (w = 1100110)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0000000</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>0101010</td>
<td>5</td>
<td>1</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>1010101</td>
<td>2</td>
<td>6</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>1110000</td>
<td>3</td>
<td>5</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>1011010</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>0100101</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>0001111</td>
<td>4</td>
<td>2</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>1111111</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

(a) Decode \(w = 0110101\) into 0100101.

(b) Decode \(w = 0101110\) into 0101010.
(c) In this case, there is a tie. There are two nearest neighbors to \( w \): 1010101 and 1011010 (at distance 2). Decode \( w \) into one of them at random. (In this case, at least two errors occurred; so there is no guarantee that the code will yield a correct decision, as indicated by the tie).

(d) Here also, there is a tie. There are four nearest neighbors to \( w \) (at distance 3): 0101010, 1110000, 0100101 and 1111111. Decode \( w \) into one of them at random.

Problem 3:

- \( C_1 \) is a \((5,2)\) group code with

\[
G_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}
\]

Its codewords \( c \) can be obtained from the operation \( c = uG \), for each message word \( u \in B^2 \). This yields

\[ C_1 = \{00000, 00111, 11110, 11001\} \]

The minimum distance \( d_{\min} = 3 \). So \( C_1 \) can detect up to two errors per codeword. It also can correct one error per codeword. (Note that \( C_1 \) is not a systematic group code; however it can be shown that we can always obtain a systematic group code \( C \) from \( C_1 \) by performing elementary row operations and column permutations on \( G_1 \). The systematic code \( C \) will have the same minimum distance as \( C_1 \), and thus yield a similar performance).

- \( C_2 \) is a \((7,3)\) systematic group code with

\[
G_2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}
\]

Thus

\[ C_2 = \{0000000, 0010111, 0101011, 0111100, 1001101, 1011010, 1100110, 1110001\} \]

The minimum distance \( d_{\min} = 4 \). So \( C_2 \) can detect three errors per codeword; it also can correct one error per codeword.
Problem 4:
The (6,3) group code with
\[
G = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix},
\]
has the following codebook:
\[
C = \{000000, 001001, 010110, 011111, 100101, 101100, 110011, 111010\}.
\]
The code has \(2^6 - 3 = 8\) cosets. The coset decoding table is as follows.

<table>
<thead>
<tr>
<th>(C)</th>
<th>000000</th>
<th>001001</th>
<th>010110</th>
<th>011111</th>
<th>100101</th>
<th>101100</th>
<th>110011</th>
<th>111010</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C + z_1)</td>
<td>000001</td>
<td>001000</td>
<td>010111</td>
<td>011110</td>
<td>100100</td>
<td>101101</td>
<td>110010</td>
<td>111011</td>
</tr>
<tr>
<td>(C + z_2)</td>
<td>000010</td>
<td>001011</td>
<td>010100</td>
<td>011101</td>
<td>100011</td>
<td>101101</td>
<td>110001</td>
<td>111000</td>
</tr>
<tr>
<td>(C + z_3)</td>
<td>000100</td>
<td>001101</td>
<td>010010</td>
<td>011011</td>
<td>100001</td>
<td>101000</td>
<td>110111</td>
<td>111110</td>
</tr>
<tr>
<td>(C + z_4)</td>
<td>010000</td>
<td>011001</td>
<td>000110</td>
<td>001111</td>
<td>110101</td>
<td>111101</td>
<td>100011</td>
<td>101010</td>
</tr>
<tr>
<td>(C + z_5)</td>
<td>100000</td>
<td>101001</td>
<td>110110</td>
<td>111111</td>
<td>000101</td>
<td>001100</td>
<td>010011</td>
<td>011010</td>
</tr>
<tr>
<td>(C + z_6)</td>
<td>000111</td>
<td>001010</td>
<td>010101</td>
<td>011100</td>
<td>100110</td>
<td>101111</td>
<td>110000</td>
<td>111001</td>
</tr>
<tr>
<td>(C + z_7)</td>
<td>011000</td>
<td>010001</td>
<td>001110</td>
<td>000111</td>
<td>111101</td>
<td>110100</td>
<td>101011</td>
<td>100010</td>
</tr>
</tbody>
</table>

The words in the second column are the coset leaders.

Problem 5:
(a) The generator matrix \(G\) of this (6,3) code is:
\[
G = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1
\end{bmatrix}.
\]

(b) The code's minimum distance is \(d = 3\). Thus the code can detect up to two errors per codeword, and it can correct one error per codeword.
(d) The syndrome decoding table is as follows.

<table>
<thead>
<tr>
<th>Syndrome $s = zH$</th>
<th>000</th>
<th>001</th>
<th>010</th>
<th>100</th>
<th>011</th>
<th>110</th>
<th>101</th>
<th>111</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coset leader $z$</td>
<td>000000</td>
<td>000001</td>
<td>000010</td>
<td>000100</td>
<td>001000</td>
<td>010000</td>
<td>100000</td>
<td>010001</td>
</tr>
</tbody>
</table>

Note that other permissible candidates for the coset leader of the last syndrome ($s = 111$) are: 100010 or 001100.

Using the above decoding table, the decoded sequence will be:

111000, 110011, 011101, 011101, 101110, 000000, 010110, 100101.