Problem 1:

xEy \iff x - y \in \mathbb{Z}, \ x, y \in \mathbb{R}.

(i) If \( x \in \mathbb{R} \), then \( x - x = 0 \in \mathbb{Z} \); then \( xEx \) and \( E \) is reflexive.

(ii) For \( x, y \in \mathbb{R} \), if \( xEy \) then \( x - y \in \mathbb{Z} \); then \( -x + y \in \mathbb{Z} \). Thus \( y - x \in \mathbb{Z} \) and \( yEx \); so \( E \) is symmetric.

(iii) For \( x, y, z \in \mathbb{R} \), if \( xEy \) and \( yEz \), then \( x - y \in \mathbb{Z} \) and \( y - z \in \mathbb{Z} \). Thus \( (x-y)+(y-z) \in \mathbb{Z} \); i.e., \( x - z \in \mathbb{Z} \) and \( xEz \). So \( E \) is transitive.

Therefore \( E \) is an equivalence relation. \( \square \)

Problem 2:

For each case, it is straightforward to show that \( R \) is an equivalence relation (check it!).

(a) \( [0] = \{0\} \), and \( [1] = [-1] = \{-1, 1\} \).

(b) \( P(B) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, B\} \). So the equivalence classes are:

\[
[\emptyset] = \{\emptyset\},
\]

\[
[\{1\}] = [\{2\}] = [\{3\}] = \{\{1\}, \{2\}, \{3\}\},
\]

\[
[\{1, 2\}] = [\{1, 3\}] = [\{2, 3\}] = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\},
\]

\[
[B] = \{B\}.
\]

(c) \( \forall (x,y) \in \mathbb{R} \times \mathbb{R}, [(x,y)] = \{(x_1,y_1) \in \mathbb{R} \times \mathbb{R} : x_1^2 + y_1^2 = x^2 + y_2^2\} \). Thus the equivalence class of \( (x,y) \) is the circle centered at the origin and with radius \( \sqrt{x^2 + y^2} \).
Problem 3:

\[ A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}. \] It is straightforward to show that in each of the following cases, \( E \) is an equivalence relation (do it!).

(a) \((a, b)E(a_1, b_1) \iff ab = a_1b_1\). The equivalence classes are:

\[
\begin{align*}
[(1, 1)] &= \{(1, 1)\}, & [(2, 2)] &= \{(2, 2)\}, & [(3, 3)] &= \{(3, 3)\}, \\
[(1, 2)] &= [(2, 1)] = \{(1, 2), (2, 1)\}, \\
[(1, 3)] &= [(3, 1)] = \{(1, 3), (3, 1)\}, \\
[(2, 3)] &= [(3, 2)] = \{(2, 3), (3, 2)\}.
\end{align*}
\]

(b) \((a, b)E(a_1, b_1) \iff a - b = a_1 - b_1\). The equivalence classes are:

\[
\begin{align*}
[(1, 1)] &= [(2, 2)] = [(3, 3)] = \{(1, 1), (2, 2), (3, 3)\}, \\
[(1, 2)] &= [(2, 3)] = \{(1, 2), (2, 3)\}, \\
[(2, 1)] &= [(3, 2)] = \{(2, 1), (3, 2)\}, \\
[(1, 3)] &= \{(1, 3)\}, & [(3, 1)] &= \{(3, 1)\}.
\end{align*}
\]

Problem 4:

(a) \(\alpha\) is not a (legitimate) function, since for example \(\alpha(3 \cdot 2) = (3, 2) \neq \alpha(2 \cdot 3) = (2, 3)\).

(b) \(\alpha\) is a function.

(c) \(\alpha\) is not a function, since for example \(\alpha(-1) = \sqrt{-1} \notin \mathbb{R}\).

Problem 5:

(a) No. \( f \) is not well-defined since for example: \( f\left(\frac{1}{2}\right) = 2 \cdot 3^2 = 18 \), and \( f\left(\frac{3}{2}\right) = 2^2 \cdot 3^4 = 324 \neq 18 \) while \( \frac{1}{2} = \frac{3}{4} \). Hence the element \( \frac{1}{2} \) is not mapped to a unique image.

(b) To make \( f \) well-defined, let \( f\left(\frac{m}{n}\right) = 2^{m/n} \).

Another good candidate for a legitimate function \( f \) is as follows: given a non-negative rational number \( r \in \mathbb{Q} \), we can write \( r \) as \( m/n \) where \( m \) and \( n \) are non-negative integers (\( n \neq 0 \)) having no common factors. In this case \( f : S \rightarrow \mathbb{R} \) defined by \( f(m/n) = 2^m 3^n \) is a legitimate function.