Material: Groups and Subgroups

Readings: Sec. 7 and 8.1 (Lecture Notes) and Sec. 4.3 and 5.1 (Humphreys-Prest text)

1. Let $G$ be a group of order 4. Assume that $e, a$ and $b$ are distinct elements of $G$ and that $a^2 = e$ and $b^2 = e$. Show that $ab$ is distinct from $a, b$ and $e$ and hence $G = \{e, a, b, ab\}$. Provide also the Cayley table for this group.

2. Let $G = \{g \in \mathbb{Q} : g \neq -1\}$; that is, $G$ is the set of all rational numbers that are not equal to $-1$. Define operation $*$ on $G$ as follows: for $a, b \in G$,

$$a * b = a + b + ab.$$ 

Show that $G$ is an abelian group under the operation $*$.

3. Let $G$ be a group and define the relation of conjugacy on $G$ by $aRb$ iff there exists $g \in G$ such that $b = g^{-1}ag$. Show that this relation $R$ is an equivalence relation on $G$.

4. In each case, specify the operation under which $G$ is a group and determine whether $H$ is a subgroup of $G$.

   (a) $G = \mathbb{Z}$; and $H = \{-1, 0, 1\}$.

   (b) $G = GL(2, \mathbb{Z})$, the set of $2 \times 2$ integer-valued matrices $A$ with determinant $det(A) = \pm 1$; and

   $$H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

   (c) $G = \mathbb{Z}_6$; and $H = \{[0], [2], [4]\}$.

   (d) $G = \mathbb{Z} \times \mathbb{Z}$; and $H = \{(m, k) \in G : m + k \text{ is even}\}$.

5. Let $g$ and $x$ be elements of a group $G$. 

(a) Show (by induction) that for all positive integers $k$,

$$(g^{-1}xg)^k = g^{-1}x^k g.$$ 

(b) Use part (a) to deduce that $x$ and $g^{-1}xg$ have the same order. [Hint: Let $d = |x|$ and $m = |g^{-1}xg|$ and show that $m|d$ and that $d|m$.]

Recommended Practice Problems: (Do not hand in)

1. Page 183, # 1. (Humphreys-Prest, 2nd Ed.)
2. Page 183, # 3. (Humphreys-Prest, 2nd Ed.)
3. Page 183, # 4. (Humphreys-Prest, 2nd Ed.)
4. Page 184, # 5. (Humphreys-Prest, 2nd Ed.)
5. Page 184, # 8. (Humphreys-Prest, 2nd Ed.)
6. Page 211, # 1. (Humphreys-Prest, 2nd Ed.)
7. Page 211, # 2 (Humphreys-Prest, 2nd Ed.)
8. Show that a group $G$ is abelian if $(gh)^3 = g^3h^3$, $(gh)^4 = g^4h^4$ and $(gh)^5 = g^5h^5$ for all $g$ and $h$ in $G$.
9. Let $H$, $K$ and $N$ be subgroups of a group $G$ and assume that $H \subseteq N$.
   (a) Show that $(HK) \cap N = H(K \cap N)$.
   (b) If both $H \cap K = N \cap K$ and $HK = NK$ hold, show that $H = N$. (Hint: First observe that $N \subseteq NK$ and that $N = (HK) \cap N$.)