1. Let $G = \mathbb{R} \times \mathbb{R}$ with addition $(x, y) + (x', y') = (x + x', y + y')$. Let $H$ be the line $y = mx$ through the origin: $H = \{(x, mx) : x \in \mathbb{R}\}$, where $m$ is a fixed real number. Show that $H$ is a subgroup of $G$ and describe the cosets $H + (a, b)$ geometrically.

2. Show that for any group element $g$, $|g| = |g^{-1}|$. [Hint: Consider two cases: $|g| = \infty$, and $|g| = n$ is finite. Use also the fact that when $|g|$ is finite, $g^k = e \iff |g|$ divides $k$.]

3. Consider the group $\mathbb{Z}_2$ with addition modulo 2, and the group $\mathbb{Z}_3$ with addition modulo 3. Construct the Cayley table for the group $\mathbb{Z}_2 \times \mathbb{Z}_3$ and show that it is cyclic.

4. Let $a^k = b^k$ in a group $G$, where $k$ is some integer. If $|a| = m$ and $|b| = n$, where $m$ and $n$ are relatively prime, show that $mn|k$. [Hint: Use Corollary 4 of Lagrange’s theorem and show that both $m$ and $n$ divide $k$.]

5. Let $G = \langle x \rangle$ be a cyclic group generated by $x$. Fix a positive integer $k$.

   (a) Show that the set $\langle x^k \rangle$ is a subgroup of $G$.

   (b) If $x$ has finite order $n$, show that $\langle x^k \rangle = \langle x^d \rangle$ and hence has $n/d$ elements, where $d = gcd(k, n)$. 


Recommended Practice Problems: (Do not hand in)

1. Page 219, # 4. (Humphreys-Prest, 2nd Ed.)
2. Page 230, # 4. (Humphreys-Prest, 2nd Ed.)
3. Page 230, # 5. (Humphreys-Prest, 2nd Ed.)
4. Page 230, # 6. (Humphreys-Prest, 2nd Ed.)
5. Page 230, # 7 (Humphreys-Prest, 2nd Ed.) [Recall that \( G_n \) is the set of invertible congruence classes in \( \mathbb{Z}_n \), see the definition on p. 47 and p. 172 (Example 3).]
6. In each case, find the right and left cosets in \( G \) of the subgroups \( H \) and \( K \).
   (a) \( G = \mathbb{Z}; H = 2\mathbb{Z} \) and \( K = 3\mathbb{Z} \).
   (b) \( G = \mathbb{Z}_{12}; H = 3\mathbb{Z}_{12} \) and \( K = 2\mathbb{Z}_{12} \).
   (c) \( G = \{e, a, a^2, a^3, b, ba, ba^2, ba^3\} \) where \(|a| = 4, |b| = 2\) and \( ab = ba^3; H = \langle a^2 \rangle \) and \( K = \langle b \rangle \).
7. Consider the group \( \mathbb{Z}_n = \{[0], [1], \ldots, [n - 1]\} \), with the addition (modulo \( n \)) operation. Show that it is cyclic.
8. (a) If \( G = \langle a \rangle \) and \(|a| = 30\), find the index of \( \langle a^6 \rangle \) in \( G \).
   (b) Let \( G = \langle a \rangle \) and \(|a| = n\). If \( d > 0 \) and \( d|n \), find the index of \( \langle a^d \rangle \) in \( G \).