1. Suppose that group $G$ has subgroups of orders 45 and 75. If $|G| < 400$, determine $|G|$. 

*Hint:* Given two integers $a$ and $b$, then the least common multiple of $a$ and $b$ – denoted by $\text{lcm}(a, b)$ – is the positive integer $m$ that satisfies: (i) $m$ is a multiple of both $a$ and $b$, and (ii) If $l$ is a multiple of both $a$ and $b$, then $l$ is a multiple of $m$.

2. Show that for any group element $g$, $|g| = |g^{-1}|$. [Hint: Consider two cases: $|g| = \infty$, and $|g| = n$ is finite and use the fact that if $|g|$ is finite, then $g^k = e \iff |g|$ divides $k$.]

3. Let $G$ be a group of order $n$ and let $m$ be an integer that is relatively prime to $n$.

   (a) If $g^m = e$ in $G$, show that $g = e$.

   (b) Show that each $g \in G$ has an $m$th root, that is, that $g = a^m$ for some $a \in G$.

4. If $H$ and $K$ are subgroups of a group and $|H|$ is prime, show that either $H \subseteq K$ or $H \cap K = \{e\}$.

5. Consider the group $\text{GL}(2, \mathbb{R})$, which is the set of $2 \times 2$ invertible real-valued matrices.

   (a) Show that

   $$H = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \text{ such that } n \in \mathbb{Z} \right\}$$

   is a cyclic subgroup of $\text{GL}(2, \mathbb{R})$.

   (b) Let $A, B \in \text{GL}(2, \mathbb{R})$ be

   $$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

   Find $|A|$, $|B|$ and $|AB|$, i.e, the orders of $A$, $B$, and $AB$. 

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**Material:** Cyclic groups, cosets, Lagrange’s theorem, group quotients

**Reading:** Sec. 8.2 and 9 (Lecture Notes) and Sec. 5.2 (Humphreys-Prest text)
Recommended Practice Problems:

1. Page 219, # 4. (Humphreys-Prest, 2nd Ed.)

2. Page 230, # 4. (Humphreys-Prest, 2nd Ed.)

3. Page 230, # 5. (Humphreys-Prest, 2nd Ed.)

4. Page 230, # 6. (Humphreys-Prest, 2nd Ed.)

5. Page 230, # 7 (Humphreys-Prest, 2nd Ed.) [Recall that $G_n$ is the set of invertible congruence classes in $\mathbb{Z}_n$, see the definition on p. 47 and p. 172 (Example 3).]

6. In each case, find the right and left cosets in $G$ of the subgroups $H$ and $K$.
   (a) $G = \mathbb{Z}; H = 2\mathbb{Z}$ and $K = 3\mathbb{Z}$.
   (b) $G = \mathbb{Z}_{12}; H = 3\mathbb{Z}_{12}$ and $K = 2\mathbb{Z}_{12}$.
   (c) $G = \{e, a, a^2, a^3, b, ba, ba^2, ba^3\}$ where $|a| = 4$, $|b| = 2$ and $ab = ba^3$; $H = \langle a^2 \rangle$ and $K = \langle b \rangle$.

7. Consider the group $\mathbb{Z}_n = \{[0], [1], \ldots, [n - 1]\}$, with the addition (modulo $n$) operation. Show that it is cyclic.

8. (a) If $G = \langle a \rangle$ and $|a| = 30$, find the index of $\langle a^6 \rangle$ in $G$.
   (b) Let $G = \langle a \rangle$ and $|a| = n$. If $d > 0$ and $d|n$, find the index of $\langle a^d \rangle$ in $G$. 

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