1. Determine in each part below if the given mapping \( \alpha : G \rightarrow G' \) is a homomorphism. If so, identify its kernel \( \text{Ker}(\alpha) \). Also determine whether \( \alpha \) is injective and/or surjective.

[Recall that the kernel of a homomorphism \( \alpha : G \rightarrow G' \) is defined by \( \text{Ker}(\alpha) = \{ g \in G : \alpha(g) = e' \} \), where \( e' \) denotes the identity of group \( G' \).]

(a) \( G = \mathbb{Z} \) under addition, \( G' = \mathbb{Z}_n \), \( \alpha(a) = [a] \) for \( a \in \mathbb{Z} \).
(b) \( G \) group, \( \alpha : G \rightarrow G \) defined by \( \alpha(a) = a^{-1} \), for \( a \in G \).
(c) \( G \) Abelian group, \( \alpha : G \rightarrow G \) defined by \( \alpha(a) = a^{-1} \), for \( a \in G \).
(d) \( G \) group of all nonzero real numbers under multiplication, \( G' = \{1, -1\} \), \( \alpha(r) = 1 \) if \( r \) is positive, \( \alpha(r) = -1 \) if \( r \) is negative.
(e) \( G \) Abelian group, \( n > 1 \) a fixed integer, and \( \alpha : G \rightarrow G \) defined by \( \alpha(a) = a^n \) for \( a \in G \).

2. Let

\[
G = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.
\]

Show that \( G \) is a group under matrix multiplication and that it is isomorphic to \( \mathbb{Z} \) (i.e., \( G \cong \mathbb{Z} \)).

3. Show that

\[
G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\}
\]

is a subgroup of \( GL_2(\mathbb{Z}) \) and is isomorphic to the group \( H = \{1, -1, i, -i\} \) (with the complex multiplication operation).
Recall that $GL_2(\mathbb{Z})$ denotes the group (under matrix multiplication) of all $2 \times 2$ invertible integer-valued matrices.]

4. If $G$ is an infinite cyclic group, show that $G \cong \mathbb{Z}$.

5. Answer the following questions.

(a) Define $\alpha : G \to G$ by $\alpha(g) = g^{-1}$. Show that $\alpha$ is an isomorphism (called an automorphism) if and only if $G$ is Abelian.

(b) Let $G, G_1, H$ and $H_1$ be groups. If $G \cong G_1$ and $H \cong H_1$, show that $G \times H \cong G_1 \times H_1$. 
