MTHE 217 - Algebraic Structures with Applications  
Fall 2018  
Homework # 9

Material: Group homomorphisms and isomorphisms

Readings: Sec. 10 (Lecture Notes) and Sec. 5.3 (Humphreys-Prest text)

1. Determine in each part below if the given mapping $\alpha : G \rightarrow G'$ is a homomorphism. If so, identify its kernel $Ker(\alpha)$. Also determine whether $\alpha$ is 1-1 or onto.

[Recall that the kernel of a homomorphism $\alpha : G \rightarrow G'$ is defined by $Ker(\alpha) = \{g \in G : \alpha(g) = e'\}$, where $e'$ denotes the unity of group $G'$.

(a) $G = \mathbb{Z}$ under addition, $G' = \mathbb{Z}_n$, $\alpha(a) = [a]$ for $a \in \mathbb{Z}$.
(b) $G$ group, $\alpha : G \rightarrow G$ defined by $\alpha(a) = a^{-1}$, for $a \in G$.
(c) $G$ abelian group, $\alpha : G \rightarrow G$ defined by $\alpha(a) = a^{-1}$, for $a \in G$.
(d) $G$ group of all nonzero real numbers under multiplication, $G' = \{1, -1\}$, $\alpha(r) = 1$ if $r$ is positive, $\alpha(r) = -1$ if $r$ is negative.
(e) $G$ abelian group, $n > 1$ a fixed integer, and $\alpha : G \rightarrow G$ defined by $\alpha(a) = a^n$ for $a \in G$.

2. Let

$$ G = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}. $$

Show that $G$ is a group under matrix multiplication and that it is isomorphic to $\mathbb{Z}$ (i.e., $G \cong \mathbb{Z}$).

3. Show that

$$ G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} $$

is a subgroup of $GL_2(\mathbb{Z})$ and is isomorphic to the group $H = \{1, -1, i, -i\}$ (with the complex multiplication operation).
[Recall that $GL_2(\mathbb{Z})$ denotes the group (under matrix multiplication) of all $2 \times 2$ invertible integer-valued matrices.]

4. If $G$ is an infinite cyclic group, show that $G \cong \mathbb{Z}$.

5. Answer the following questions.

   (a) Define $\alpha : G \to G$ by $\alpha(g) = g^{-1}$. Show that $\alpha$ is an isomorphism (called an automorphism) if and only if $G$ is abelian.

   (b) Let $G, G_1, H$ and $H_1$ be groups. If $G \cong G_1$ and $H \cong H_1$, show that $G \times H \cong G_1 \times H_1$. 