1. (a) Consider the following statement: “If 6 is related to 18 and 18 is related to 72, then 6 is related to 72.”
   i. Write a formula for this statement using symbolic logic.
   ii. Find the negation of the formula.
   iii. Express the negation in words.

(b) Determine whether the following argument is valid by representing the sentences as statement forms and checking to see whether the conjunction of the assumptions logically implies the conclusion.

   Randy does not work hard. In order for Randy to pass this course, it is necessary that he work hard. Therefore, Randy will not pass this course.

2. (a) Use the fact that if two finite sets $C$ and $D$ are disjoint (i.e., $C \cap D = \emptyset$), then $|C \cup D| = |C| + |D|$, to show that for any two finite sets $A$ and $B$,

   $$|A \cup B| = |A| + |B| - |A \cap B|.$$ 

(b) Use (a) to show that for any three finite sets $A$, $B$ and $C$,

   $$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$ 

3. Assume that $\alpha : S \to T$ and $\beta : T \to U$. Consider the following statement.

   If $\alpha$ is one-to-one and $\beta$ is onto, then $\beta \alpha$ is one-to-one and onto. \hfill (⋆)

(a) Show that the Statement (⋆) is false by providing a counterexample.

(b) Write the converse of Statement (⋆) and show that it is true.

(c) Write the contrapositive of Statement (⋆). Is this contrapositive true? Justify your answer.
4. For each ordered pair \((a,b)\) of integers define a function \(\alpha_{a,b} : \mathbb{Z} \to \mathbb{Z}\) by \(\alpha_{a,b}(n) = an + b\), where \(\mathbb{Z}\) is the set of integers.

   (a) For which values of \(a\) and \(b\) is \(\alpha_{a,b}\) onto?

   (b) For which values of \(a\) and \(b\) is \(\alpha_{a,b}\) one-to-one?

5. (a) Let \(\alpha : A \to B\) be a function, and assume that \(|A| = |B|\). Show that if \(\alpha\) is one-to-one, then it is onto.

   (b) Let \(\alpha : A \to B\) and \(\beta : B \to C\) be functions. If \(\beta\alpha\) is one-to-one and \(\alpha\) is onto, show that \(\beta\) is one-to-one.

6. Let \(\alpha : S \to T\) be one-to-one, and let \(A\) and \(B\) be subsets of \(S\). Assume that \(S\), \(T\), \(A\), and \(B\) are non-empty. Show that \(\alpha(A \cap B) = \alpha(A) \cap \alpha(B)\).

7. (a) Given a set \(S = \{w, x, y, z\}\), let \(R\) be a relation on \(S\) and assume that \(wRy\) and \(zRy\). Which of the following must also be true if \(R\) is to be an equivalence relation on \(S\)?

   (i) \(yRy\)  (ii) \(yRz\)  (iii) \(wRz\)
   (iv) \(yRx\)  (v) \(xRy\)  (vi) \(zRz\)

   (b) Let \(R\) be a relation on a set \(S\). Complete each of the following statements.

   (i) \(R\) is not reflexive iff \(\ldots\)
   (ii) \(R\) is not symmetric iff \(\ldots\)
   (iii) \(R\) is not transitive iff \(\ldots\)

8. Define a relation \(E\) on the set of positive integers \(\mathbb{P}\) by

\[
aEb \text{ iff } a = b \cdot 10^k \text{ for some } k \in \mathbb{Z},
\]

where \(\mathbb{Z}\) denotes the set of integers. Show that \(E\) is an equivalence relation on \(\mathbb{P}\).

9. Use mathematical induction to show that \(3^{2n-1} + 2^{n+1}\) is a multiple of 7 for all positive integers \(n\).
10. (a) Use mathematical induction to prove that the following statement holds for all integers \( n \geq 1 \).

\[
1^2 - 2^2 + \cdots + (-1)^{n+1} n^2 = \frac{(-1)^{n+1} n(n + 1)}{2}.
\]

(b) *DeMorgan’s law*: Let \( A \) and \( B \) denote subsets of a universe \( U \). Show that

\[
(A \cup B)^c = A^c \cap B^c.
\]

(c) *Extended DeMorgan’s law*: Use induction on the integers \( n \geq 2 \) to show that if \( A_1, A_2, \ldots, A_n \) are subsets of a universe \( U \), then

\[
(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c.
\]

11. Use the division algorithm to find the \( \gcd(532, 378) \), and write it as a linear combination of 532 and 378.

12. Let \( m \) and \( n \) be integers. If \( \gcd(m, n) = 1 \), let \( d = \gcd(m + n, m - n) \). Show that \( d = 1 \) or \( d = 2 \).

13. For any integer \( n \), show that \( \gcd(5n + 3, 7n + 4) = 1 \).

14. Prove that the square of any odd number has the form \( 8k + 1 \) for some integer \( k \).

15. Let \( a \) and \( b \) be integers, not both 0. Prove that an integer \( e \) is a linear combination of \( a \) and \( b \) if and only if \( e \) is a multiple of \( \gcd(a, b) \).

16. Let \( m \) and \( n \) be relatively prime integers. Show that if \( m|k \) and \( n|k \) for some integer \( k \), then \( mn|k \).

17. Show using induction that \( n^3 + 5n \) is a multiple of 6 for all integers \( n \geq 1 \).

18. Let \( a \) and \( b \) be two integers such that \( a > b > 0 \). Let \( r \) be the remainder of the division of \( a \) by \( b \); i.e., \( a = qb + r \) where \( q \) is an integer and \( 0 \leq r < b \). Show that

\[
\gcd(a, b) = \gcd(b, r),
\]

and briefly describe an effective algorithm for computing \( \gcd(a, b) \).