STUDENT NUMBER: ______________________________

Math/Mthe 225 — Quiz 2

7 October 2019

Professor E. Kani

- This is a 35 minute test.
- No textbooks, notes, or aids other than your calculator are allowed.
- Do not turn the first page until instructed by your proctor.
- For full marks, you must show all your work and explain how you arrived at your answers, unless explicitly told to do otherwise.
- Wherever appropriate, include units in your answers.
- While waiting to start, **please fill in your name and student number above.**
- If you need more room, there is a blank page at the end of the test. **If you use these pages, you must provide clear directions to the marker**, e.g. "Continued on page 4".

Q1	Q2	Total
6	4	10

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Question 1 (6 marks).

A tank initially contains 60 gal of pure water. Brine containing containing 1 lb of salt per gallon enters the tank at 2 gal/min, and the (perfectly mixed) solution leaves the tank at 3 gal/min. Find the amount of salt in the tank after t minutes.

Solution: Let $V(t)$ denote the volume of the solution in the tank. We have $V(0) = 60$, $r_i = 2$ (gal/min) and $r_o = 3$ (gal/min), so

$$V(t) = V(0) + (r_i - r_0)t = 60 + (2 - 3)t = 60 - t.$$

Next, let $x(t)$ be the amount of salt (in lbs.) in the tank at time t. We have $x(0) = 0$ because there is no salt in the tank at $t = 0$. Moreover,

$$x'(t) = r_i c_i - r_o \frac{x(t)}{V(t)} = 2(1) - 3\frac{x(t)}{60 - t}.$$

To solve this linear DE, find an integrating factor $\rho(t)$. Here

$$\rho(t) = e^{\int \frac{3}{60-t} dt} = e^{-3\ln(60-t)} = (60 - t)^{-3}.$$

Then $(\rho(t)x(t))' = 2\rho(t) = \frac{2}{(60-t)^3}$. Integrating gives

$$\rho(t)x(t) = \int \frac{2dt}{(60 - t)^3} = \frac{1}{(60 - t)^2} + C$$

and so $x(t) = (60 - t) + \frac{C}{2}(60 - t)^3$. Since $x(0) = 0$, we have $0 = (60) + \frac{C}{2}60^3$, so $\frac{C}{2} = -\frac{60}{60^3} = \frac{-1}{3600}$, and hence

$$x(t) \; = \; (60 - t) - \frac{(60 - t)^3}{3600}.$$

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Question 2 (4 marks).

Solve the following differential equation.

$$ty^2 \frac{dy}{dt} = y^3 - t^3.$$

Solution: Since $y = 0$ is not a solution of the DE, we can divide through by ty^2 (if $t \neq 0$) to obtain $y' = \frac{y}{t} - \left(\frac{t}{y}\right)^2$. Thus, this DE is homogeneous in $v = \frac{y}{t}$. Since $y' = (vt)' = v't + v$, we get that

$$tv' + v = v - v^{-2} \quad \text{or} \quad v^2 v' = -\frac{1}{t},$$

which is a separable DE. Integrating gives

$$\frac{1}{3}v^3 = -\ln|t| + C \quad (\text{for } t \neq 0) \text{ or } \quad v^3 = -3\ln|t| + 3C.$$

Thus, $v = (-3\ln|t| + C)^{\frac{1}{3}}$, which means that

$$y = t(-3\ln|t| + C)^{\frac{1}{3}}, \quad \text{for } t \neq 0.$$

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Space for additional work. **Indicate clearly which question you are continuing if you use this space.**