

STUDENT NUMBER:

Math/Mthe 225 — Quiz 3**28 October 2019****Professor E. Kani**

- This is a 35 minute test.
- No textbooks, notes, or aids other than your calculator are allowed.
- Do not turn the first page until instructed by your proctor.
- For full marks, you must show all your work and explain how you arrived at your answers, unless explicitly told to do otherwise.
- Wherever appropriate, include units in your answers.
- While waiting to start, **please fill in your name and student number above.**
- If you need more room, there is a blank page at the end of the test. **If you use these pages, you must provide clear directions to the marker**, e.g. “Continued on page 4”.

Q1	Q2	Total
6	4	10

Question 1 (6 marks).

Verify that the following differential equation is exact, and find its implicit solution.

$$y^2 + 2xy + 2x + (x^2 + 2xy + 3y^2)y' = 0$$

Solution: Let $M(x, y) = y^2 + 2xy + 2x$ and $N(x, y) = x^2 + 2xy + 3y^2$. Then

$$\frac{\partial M}{\partial y}(x, y) = 2y + 2x + 0 \quad \text{and} \quad \frac{\partial N}{\partial x}(x, y) = 2x + 2y + 0, \quad \text{so} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

and hence the DE is exact (by a result in class). This means that there exists a function $F(x, y)$ such that

$$\frac{\partial F}{\partial x}(x, y) = M(x, y) \quad \text{and} \quad \frac{\partial F}{\partial y}(x, y) = N(x, y).$$

Integrating the first relation (with respect to x) gives

$$F(x, y) = \int M(x, y)dx = \int (y^2 + 2xy + 2x)dx = xy^2 + x^2y + x^2 + g(y).$$

If we differentiate this relation (with respect to y) and use the fact that $\frac{\partial F}{\partial y} = N$, then we get

$$x^2 + 2xy + 3y^2 = N = \frac{\partial F}{\partial y} = 2xy + x^2 + g'(y),$$

so we obtain that $g'(y) = 3y^2$. Integrating this yields $g(y) = y^3 + C$, so by the above

$$F(x, y) = xy^2 + x^2y + x^2 + g(y) = xy^2 + x^2y + x^2 + y^3 + C.$$

Thus, the implicit solution of the DE is $xy^2 + x^2y + x^2 + y^3 + C = 0$. We can also write this as

$$xy^2 + x^2y + x^2 + y^3 = C \quad (\text{for a different constant } C.)$$

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Question 2 (4 marks).

Are the following functions y_1 , y_2 and y_3 linearly independent on the real line? Justify your answer.

$$y_1 = e^x, \quad y_2 = e^{2x}, \quad y_3 = x.$$

Solution: We can use the Wronskian test to check for independence. Expanding the determinant along the last row gives

$$\begin{aligned} W(y_1, y_2, y_3) &= \det \begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{pmatrix} \\ &= y_1'' \det \begin{pmatrix} y_2 & y_3 \\ y_2' & y_3' \end{pmatrix} - y_2'' \det \begin{pmatrix} y_1 & y_3 \\ y_1' & y_3' \end{pmatrix} + y_3'' \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \\ &= e^x \det \begin{pmatrix} e^{2x} & x \\ 2e^{2x} & 1 \end{pmatrix} - 4e^{2x} \det \begin{pmatrix} e^x & x \\ e^x & 1 \end{pmatrix} + 0 \det(\dots) \\ &= e^x(e^{2x} - 2xe^x) - 4e^{2x}(e^x - xe^x) = e^{3x}(1 - 2x - 4 + 4x) = e^{3x}(2x - 3). \end{aligned}$$

Alternately, if we use the formula for a 3×3 determinant, then we get:

$$\begin{aligned} W(y_1, y_2, y_3) &= \det \begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{pmatrix} = \det \begin{pmatrix} e^x & e^{2x} & x \\ e^x & 2e^{2x} & 1 \\ e^x & 4e^{2x} & 0 \end{pmatrix} \\ &= 0 + e^{2x}(1)e^x + xe^x(4e^{2x}) - x(2e^{2x})e^x - 0 - e^x(1)(4e^{2x}) \\ &= e^{3x}(1 + 4x - 2x - 4) = e^{3x}(2x - 3). \end{aligned}$$

Thus $W(y_1, y_2, y_3)(x) = e^{3x}(2x - 3)$. For $x = 0$ we get $W(y_1, y_2, y_3)(0) = e^0(-3) = -3 \neq 0$, so the functions are linearly independent.

Notes: 1) It is important to verify that $W(y_1, y_2, y_3)(x) \neq 0$ for at least one point $x \in \mathbb{R}$. It is not enough to say $W(y_1, y_2, y_3)(x) \neq 0$.

2) Since $W(y_1, y_2, y_3)(\frac{3}{2}) = 0$, it follows from our results in class that these three functions cannot be solutions of a 3^{rd} order linear homogeneous DE on the real line. They are, however, solutions of a 4^{th} order linear homogeneous DE (with constant coefficients).

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Space for additional work. **Indicate clearly which question you are continuing if you use this space.**