STUDENT NUMBER: ____________________

Math/Mthe 225 — Quiz 4

11 November 2019

Professor E. Kani

- This is a 35 minute test.
- No textbooks, notes, or aids other than your calculator are allowed.
- Do not turn the first page until instructed by your proctor.
- For full marks, you must show all your work and explain how you arrived at your answers, unless explicitly told to do otherwise.
- Wherever appropriate, include units in your answers.
- While waiting to start, **please fill in your name and student number above.**
- If you need more room, there is a blank page at the end of the test. **If you use these pages, you must provide clear directions to the marker**, e.g. "Continued on page 4".

Q1	Q2	Total
4	6	10

Question 1 (4 marks).

Find the general solution of the following differential equation.

$$y^{(4)} + 4y^{(3)} + 8y^{(2)} \;=\; 0.$$

Solution. The characteristic equation is $X^4 + 4X^3 + 8X^2 = 0$. Since $X^4 + 4X^3 + 8X^2 = X^2(X^2 + 4X + 8)$, we see by the quadratic formula that the second quadratic factor has roots

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2} = \frac{-4 \pm \sqrt{4^2 - 4(8)}}{2} = \frac{-4 \pm \sqrt{-16}}{2} = -2 \pm 2i.$$

Thus, the roots are 0 (of multiplicity 2) and $-1 \pm 2i$. The part of the solution of the DE corresponding to 0 is $c_1 e^{0x} + c_2 x e^{0x} = c_1 + x c_2$, and the solutions corresponding to the complex conjugate pair $-1 \pm 2i$ are $c_3 e^{-x} \cos(2x) + c_4 e^{-x} \sin(2x)$. Thus, the general solution is

$$y(x) \;=\; c_1 + c_2 x + c_3 e^{-2x} \cos(2x) + c_4^{-2x} \sin(2x).$$

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Question 2 (6 marks).

Solve the differential equation

$$y'' - 4y' + 4y \ = \ 4 + 2e^{2x}.$$

Solution. Step 1: Find the complementary solution $y_c(x)$.
The characteristic equation of the associated homogeneous DE is $X^2 - 4X + 4 = 0$. Since $X^2 - 4X + 4 = (X-2)^2$, we see that this has a 2 as a double root, and so the complementary solution is

$$y_c(x) = c_1 e^{2x} + c_2 x e^{2x} = (c_1 + c_2 x)e^{2x}.$$

Step 2: Find a particular solution $y_p(x)$.
We use the method of undetermined coefficients. Since e^{2x} appears in $y_c(x)$, we need to apply Rule 2 to obtain a trial solution. Since xe^{2x} also appears (or since the root 2 has multiplicity 2), Rule 2 tells us to take $y_p(x) = A + Bx^2e^{2x}$ as our trial solution. Then

$$\begin{aligned} y_p'(x) &= B\left(2xe^{2x} + x^2(2e^{2x})\right) = B\left(2x + 2x^2\right)e^{2x}, \\ y_p''(x) &= B\left(2+4x)e^{2x} + \left((2x+2x^2)(2e^{2x})\right)\right) \ = \ B\left(2 + 8x + 4x^2\right)e^{2x}. \end{aligned}$$

Substituting this into the DE yields

$$\begin{aligned} 4 + 2e^{2x} &= B\left(2 + 8x + 4x^2\right)e^{2x} - 4\left(B\left(2x + 2x^2\right)e^{2x}\right) + 4(A + Bx^2e^{2x}) \\ &= 4A + Be^{2x}(2 + 8x + 4x^2 - 8x - 8x^2 + 4x^2) = 4A + 2Be^{2x} \end{aligned}$$

so $4A = 4$ and $2B = 2$, or $A = B = 1$. Thus, $y_p(x) = 1 + x^2e^{-x}$ is a particular solution, and hence the solution is

$$y(x) \ = \ y_p(x) + y_c(x) = (c_1 + c_2 x)e^{2x} + 1 + x^2e^{2x}.$$

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Space for additional work. **Indicate clearly which question you are continuing if you use this space.**