STUDENT NUMBER: ______________________________

Math/Mthe 225 — Quiz 5

25 November 2019

Professor E. Kani

- This is a 35 minute test.
- No textbooks, notes, or aids other than your calculator are allowed.
- Do not turn the first page until instructed by your proctor.
- For full marks, you must show all your work and explain how you arrived at your answers, unless explicitly told to do otherwise.
- Wherever appropriate, include units in your answers.
- While waiting to start, **please fill in your name and student number above.**
- If you need more room, there is a blank page at the end of the test. **If you use these pages, you must provide clear directions to the marker**, e.g. "Continued on page 4".

Q1	Q2	Total
4	6	10

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Question 1 (4 marks).

Find the inverse Laplace transform of $F(s) \;=\; \dfrac{5}{s(s^2+4s+5)}$.

Solution. Since $s^2+4s+5=(s+2)^2+1$ is irreducible, the partial fraction decomposition of $F(s)$ has the form

$$F(s) \;=\; \frac{A}{s}+\frac{Bs+C}{s^2+4s+5} \;=\; \frac{A(s^2+4s+5)+s(Bs+C)}{s(s^2+4s+5)} \;=\; \frac{(A+B)s^2+(4A+C)s+5A}{s(s^2+4s+5)}$$

and so $(A+B)s^2+(4A+C)s+5A=5$. This gives the equations

$$A+B \;=\; 0, \quad 4A+C \;=\; 0, \quad \text{and} \quad 5A \;=\; 5$$

Thus, $A=\frac{5}{5}=1$, $B=-A=-1$ and $C=-4A=-4$, and hence

$$F(s) \;=\; \frac{1}{s}-\frac{s+4}{(s+2)^2+1} \;=\; \frac{1}{s}-\frac{s+2}{(s+2)^2+1^2}-2\frac{1}{(s+2)^2+1^2}$$

Applying the inverse Laplace transform $\mathcal{L}^{-1}\{..\}$ and using the tables (and the first translation theorem) yields

$$\mathcal{L}^{-1}\{F(s)\}(t) \;=\; 1-e^{-2t}\cos(t)-2e^{-2t}\sin(t).$$

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Question 2 (6 marks).

Use Laplace transforms to solve the following initial value problem:

$$x'' + 4x \; = \; 8e^{2t}, \quad x(0) = 0,\, x'(0) = 0.$$

Solution. Put $X(s) = \mathcal{L}\left\{x(t)\right\}(s)$. Then by the derivative formula we have $\mathcal{L}\left\{x''(t)\right\}(s) = s^2X(s) - 0 - 0 = s^2X(s)$. Thus, applying the Laplace transform to both sides gives

$$(s^2+4)X(s) \; = \; 8\mathcal{L}\left\{e^{2t}\right\} \; = \; \frac{8}{s-2} \quad \Rightarrow \quad X(s) = \frac{8}{(s-2)(s^2+4)}$$

Since $s^2 + 4$ is irreducible, the partial fraction decomposition of $X(s)$ has the form

$$X(s) \; = \; \frac{A}{s-2} + \frac{Bs+C}{s^2+4} \; = \; \frac{A(s^2+4)+(s-2)(Bs+C)}{(s-2)(s^2+4)} \; = \; \frac{(A+B)s^2+(C-2B)s+4A-2C}{(s-2)(s^2+4)}$$

and so $(A+B)s^2 + (C-2A)s + 4A - 2C = 5$. This gives the equations

$$A+B \; = \; 0, \quad C-2B \; = \; 0, \quad \text{and} \quad 4A-2C \; = \; 8$$

Substituting $C = 2B$ into the last equation gives $4A - 4B = 8$ or $A - B = 2$, and hence, since $B = -A$ by the first equation, we get $2A = 2$ or $A = 1$. Thus $B = -A = -1$, and $C = 2B = -2$. Thus

$$X(s) \; = \; \frac{1}{s-2} - \frac{s}{s^2+2^2} - \frac{2}{s^2+2^2}.$$

Applying the inverse transform and using the tables gives

$$x(t) \; = \; e^{2t} - \cos(2t) - \sin(2t).$$

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Space for additional work. **Indicate clearly which question you are continuing if you use this space.**

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Table of Laplace Transforms (DO NOT REMOVE FROM BOOKLET)

- $\mathcal{L}\{1\} = \dfrac{1}{s}$,
- $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$, for $n \in \mathbb{N}$.
- $\mathcal{L}\{t^a\} = \dfrac{\Gamma(a+1)}{s^{n+1}}$, for any $a > -1$.
- $\mathcal{L}\left\{e^{at}\right\} = \dfrac{1}{s-a}$, for any $a \in \mathbb{R}$.
- $\mathcal{L}\{\sin(kt)\} = \dfrac{k}{s^2+k^2}$, for any $k \in \mathbb{R}$.
- $\mathcal{L}\{\cos(kt)\} = \dfrac{s}{s^2+k^2}$, for any $k \in \mathbb{R}$.
- $\mathcal{L}\{\sinh(kt)\} = \dfrac{k}{s^2-k^2}$, for any $k \in \mathbb{R}$.
- $\mathcal{L}\{\cosh(kt)\} = \dfrac{s}{s^2-k^2}$, for any $k \in \mathbb{R}$.
- $\mathcal{L}\left\{x'(t)\right\} = sX(s) - x(0)$, where $X(s) = \mathcal{L}\{x(t)\}$.
- $\mathcal{L}\left\{x''(t)\right\} = s^2X(s) - sx(0) - x'(0)$, where $X(s) = \mathcal{L}\{x(t)\}$.
- $\mathcal{L}\left\{x^{(n)}(t)\right\} = s^nX(s) - s^{n-1}x(0) - s^{n-2}x'(0) - \ldots - x^{(n-1)}(0)$, if $n \in \mathbb{N}$ and $X(s) = \mathcal{L}\{x(t)\}$.
- $\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a)$, where $F(s) = \mathcal{L}\{f(t)\}$.
- $\mathcal{L}\{u_a(t)f(t-a)\} = e^{-as}F(s)$, where $F(s) = \mathcal{L}\{f(t)\}$ and $u_a(t) = \begin{cases} 1 & \text{if } t \geq a, \\ 0 & \text{if } t < a. \end{cases}$