Ex. 3: Solve
$$y' = \sin(x) - \frac{y}{2}.$$

Sol'n: Adding $y/2$ to both sides gives
$$(*) \qquad y' + \tfrac{1}{2}y = \sin(x),$$
so we see that this is linear with $P(x) = \frac{1}{2}$, and $Q(x) = \sin(x)$. We follow the method:

1) Compute $\tilde{P}(x) = \int P(x)dx = \int \frac{1}{2}dx = \frac{x}{2} (+C)$.

2) Thus, an integrating factor is
$$\rho(x) = e^{\tilde{P}(x)} = e^{x/2}.$$

Multiplying $(*)$ by $\rho(x)$ gives
$$(\rho(x)y(x))' = \rho(x)Q(x) = e^{x/2}\sin(x).$$

3) Integrating:
$$\rho(x)y(x) = \int \rho(x)Q(x)dx = \int e^{x/2}\sin(x)dx.$$

Thus, we have to evaluate the integral
$$I := \int e^{x/2}\sin(x)dx.$$

For this, we use <u>integration by parts</u>, which is given by the formula
$$(**) \qquad \int uv' = uv - \int vu'.$$

Here:

Take $u = \sin(x)$, so $u' = \cos(x)$,

$v = 2e^{x/2}$, so $v' = 2(\frac{1}{2}e^{x/2}) = e^{x/2}$.

Then by (**)

$$(***) \quad I = \int e^{x/2}\sin(x)\,dx = \sin(x)(2e^{x/2}) - \int (2e^{x/2})\cos(x)\,dx$$
$$= 2e^{x/2}\sin(x) - 2\underbrace{\int e^{x/2}\cos(x)\,dx}_{J}$$

Integrating J by parts with

$u = \cos(x) \Rightarrow u' = -\sin(x)$

$v = 2e^{x/2} \Rightarrow v' = e^{x/2}$

gives

$$J = 2e^{x/2}\cos(x) - \int 2e^{x/2}(-\sin(x))$$
$$= 2e^{x/2}\cos(x) + 2\underbrace{\int e^{x/2}\sin(x)}_{I}$$

Substituting this in (***) yields

$$I = 2e^{x/2}\sin(x) - 2J$$
$$= 2e^{x/2}\sin(x) - 2(2e^{x/2}\cos(x) + 2I)$$
$$= 2e^{x/2}(\sin(x) - 2\cos(x)) - 4I.$$

Adding $4I$ to both sides gives

$$5I = 2e^{x/2}(\sin(x) - 2\cos(x)),$$
$$I = \frac{2}{5}e^{x/2}(\sin(x) - 2\cos(x)).$$

We thus have

$$\rho(x)\,y(x) = I(x) + C$$

and so

$$\begin{aligned} y(x) &= I(x)e^{-x/2} + Ce^{-x/2} \\ &= \tfrac{2}{5}(\sin(x) - 2\cos(x)) + Ce^{-x/2}, \end{aligned}$$

is the general solution of the DE.

Note: $\int e^{at}\sin(bt)\,dt$ is given in formula 49. on the inside cover of the text.