Separation of Estimation and Control for Discrete Time Systems

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Summarizing the Paper
A separation structure refers to a situation where the controller first estimates the state, and then applies its control action, by regarding the estimate as the state itself. This paper is an attempt to clear up some of the confusion surrounding separation and to summarize some of the results that can be associated with the vague concept of “separation” between Estimation and control.

This paper is restricted to discrete time systems with finite horizons, since as soon as inputs or outputs or both are updated at intervals one is reduced in effect to a discrete time situation.

Problem Description
Consider a system operating for T time steps. Observations are made at each step from M observation posts and control inputs are applied at each step from K control stations. Here $T > 0$, $K > 0$, $M \geq 0$ are given integers.

The operation of this system can be described with the variables $x_t, u_t^k, y_t^m$ denoting the state transition, control, and observation vectors respectively. The subscript $t$ denotes time, and the superscript denotes the particular station. The uncertainties of the system are modeled by independent random variables called the “primitive random variables”:

$$x_0, v_t, w_t^m \ (t = 1, \ldots, T; m = 1, \ldots, M).$$

The dynamics and the observations of the system are given by,

$$x_t = f_t(x_{t-1}, u_t^1, \ldots, u_t^K) \quad k = 1, \ldots, K \text{ and } t = 1, \ldots, T$$

$$y_t^m = g_t^m(x_{t-1}, w_t^m) \quad m = 1, \ldots, M \text{ and } t = 1, \ldots, T$$

The cost is given by the expression,
\[ \sum_{t=1}^{T} h_t(x_t, u_t^1, \ldots, u_t^K) \]

Control laws are the possible ways in which the input \( u_t^k \) can be generated. To do this one must specify the data available as arguments, ranges and restrictions on the functional form of the law.

The specification of the data available is the information pattern, which is the assignment to each index pair \( (t, m) \) for observation \( Y_t \), or to the index pair \( (t, k) \) for control \( U_{t-1} \) of a data basis at time \( t \). Define the following sets of pairs of indices. For \( t = 1, \ldots, T \),

\[ Y_t = \{(\tau, m) | \tau = 1, \ldots, t; m = 1, \ldots, M \}. \]

For \( t = 2, \ldots, T + 1 \)

\[ U_t = \{(\tau, k) | \tau = 1, \ldots, t-1; k = 1, \ldots, K \}. \]

Also,

\[ U_1 = \emptyset \]

The control equations for this system can be shown as

\[ u_t^k = \gamma^k_t \left( \cdots, y_{t+1}^\mu, \cdots, u_0^k, \cdots \right) \]

\[ = \gamma^k_t (y_{t,k}, u_{t,k}) \]

Where, for each \( (t,k) \) in \( U_{t+1} \), \( \gamma^k_t \) is the control law for that index pair are the arguments specified by the information pattern.
**Topics in the Study of Information Patterns**

**Some Types of Information Patterns**

*Perfect recall* is when any data that the control station had at some time remains available to it at any later time. Therefore, given an information pattern, and a control station $k$, either $T = 1$ or else one has $Y_{t,k} \subseteq Y_{t+1,k}$, $U_{t,k} \subseteq U_{t+1,k}$ for $t = 1, \ldots, T - 1$. If a station does not record all of the information it can still have perfect recall if it can reconstitute the forgotten controls from what it remembers.

An information pattern is called *classical*, if all stations receive the same information pattern (independent of subscript $k$) and have perfect recall.

Let $M = K$ (hence $Y_t = U_{t+1}$) and $n > 0$. Let $Y_{t,k}$ consist of all pairs $(\theta, \mu)$ with $\theta \leq t - 1$, $1 \leq \mu \leq K$ and the pairs $(\theta, K)$ with $t - n < \theta \leq t$. Let $U_{t,k}$ contain the same pairs except for those $\theta = t$. This is called a *delayed sharing patterns*. This means that stations share their data with a delay of $n$ time steps.

**Field Basis and Conditioning**

A triple $(Y, U, L)$ with $Y$ a subset of $Y_t$ and $U$, $L$ subsets of $U_t$ is called a *field basis* (at time $t$) if for any two designs $\gamma, \overline{\gamma}$ in $\Gamma$ the relation $\gamma_L = \overline{\gamma}_L$, that is $\gamma_{\theta}^k(\cdot) \equiv \overline{\gamma}_{\theta}^k(\cdot)$ for $(\theta, k)$ in $L$, implies that the fields $F(Y, U; \gamma)$ and $F(Y, U; \overline{\gamma})$ are the same.

A field basis simply means knowledge of just those control laws designated by $L$ (a subset of $U_t$) is sufficient to determine unambiguously the field generated by the variables in $(Y, U)$.

Answers the question “How much information about the control laws is required to produce an estimation for the state $x_{t-1}$ given the data basis $(Y, U)$?” Therefore, a conditioning basis is a set of indices on $\gamma_t^k$. 
Equivalences and Substitutions
Substitutions between data are based on certain equivalences. For any design \( \gamma \) the dependence on \( u \) can be eliminated by recursive substitution in the system equations, this process turns the control variables into random variables by the substitution

\[
 u = S(w; \gamma)
\]

This process allows every variable of the form \( z = f(w, u) \) become a parametric family of random variables, parameterized by the design \( \gamma \),

\[
 z = f(\omega, S(\omega; \gamma)) \equiv g(\omega; \gamma)
\]

Two designs \( \gamma, \gamma^* \) are called equivalent when \( S(\omega; \gamma) = S(\omega; \gamma^*) \) for almost all \( \omega \). This implies that for any system variable, such as \( z \),

\[
 g(\omega; \gamma) = g(\omega; \gamma^*)
\]

Two information patterns \((Y_{t,k}, U_{t,k})\) \((Y^*_{t,k}, U^*_{t,k})\) are called equivalent when for any design feasible with the first pattern there is an equivalent design feasible with the second pattern and vice versa.

**Assertion 1:** If, for every \((t, k)\) \((Y_{t,k}, U_{t,k}, \phi)\) is a field basis, then the given feedback control problem is equivalent to a feedforward control problem.

Note that a feedforward control problem, is one in which there is not feedback in the system, in that is passes a controlling signal from a source to the systems external operator.

**Assertion 2:** This is a more common form of equivalence. Suppose that for some pair \((t, k)\) there is a function \( \phi \) such that, for all \( \omega \) and \( \gamma \),
\[ (y_{t,k}, u_{U,t,k}) = \phi(y_{U,t,k}, y_{U,t,k}) \]

with \( Y \subset Y_{t,k}, \ U \subset U_{t,k} \). Then the given pattern is equivalent to the one in which \((Y_{t,k}, U_{t,k})\) is replaced by \((Y, U)\). This can be seen from the following substitution,

\[
\gamma^{*k}_{t}(y_{Y,t}, u_{U}) = \gamma^{*k}_{t}(\phi(y_{Y}, u_{U}, y_{U}, u_{U,^t})) \\
\gamma^{*k}_{t} = \gamma^{*k}_{\tau}, \text{ for } (\tau, \kappa) \neq (t, k)
\]

noting that

\[ \gamma^{*}_{U,t} = \gamma_{U,t} \]

**Assertion 3:** Suppose \((Y, U, L)\) is a conditioning basis for \( z \) and one has 
\((t, k) \in U \cap L, \ Y_{t,k} \subset Y, \ U_{t,k} \subset U \). Then \((Y, U, L), (Y, U - \{(t, k)\}, L)\), and \((Y, U, L - \{(t, k)\})\) are equivalent conditioning bases for \( z \).

**Assertion 4:** For an \( n \)-step delayed sharing pattern and any \((t, k) \in U_{T+1}\) the triple \((Y_{t,k}, U_{t,k}, L_{t,k})\) and the triple \((\bigcap_{k=1}^{K} Y_{t,k}, \bigcap_{k=1}^{K} U_{t,k}, \emptyset)\) are both conditioning bases for \( x_{t-n} \).

**Some Results Related to Separation**

**The General Separation for the Strictly Classical Pattern**

Consider a strictly classical information pattern. Note that a strictly classical information pattern is a special case of the \( n \)-step delay sharing pattern, where the system has a single controller and observer. Therefore we can apply Assertion 4, which means that there exists a conditional distribution of \( x_{t-n} \) given \((y_{1},..., y_{t}, u_{1},..., u_{t-1})\). We represent this distribution as a function \( F_{t}(y_{1},..., y_{t}, u_{1},..., u_{t-1}) \). Evaluating this function is called filtering. Filtering is completely independent of the design \( \gamma \).
**Assertion 5:** In a strictly classical problem, a separation structure exists. In other words, there is no loss if we let the design be of the form $\gamma_t = \phi_t \circ F_t$.

This is a very useful result as the problem of determining $\phi_t$ is Markovian. Note however that finding an optimal $\phi_t$ depends on the choice of filter, so the filtering algorithm must be determined first.

**Linear Gaussian Systems: Filtering**

Consider a linear system with Gaussian noise and an n-step delayed sharing pattern. At time $t$, the data that has already been shared and is available to all controllers is designated by $\bigcap_{k=1}^K Y_{t,k} \cap \bigcap_{k=1}^K U_{t,k}$. In this type of system, each controller shares its information with an n-step delay, so $\bigcap_{k=1}^K Y_{t,k} \cap \bigcap_{k=1}^K U_{t,k} = (Y_{t-n,k}, U_{t-n,k}) \forall 1 \leq k \leq K$.

**Assertion 6:** In the system described above, for any $(t,k)$ in $U_{T+1}$ the conditional distribution of $x_{t-n}$ given the data designated by $\bigcap_{k=1}^K Y_{t,k} \cap \bigcap_{k=1}^K U_{t,k}$ has a Gaussian version with covariance independent of the data and mean affine in the data (in other words, the mean is some affine function $F_t(y_1,\ldots,y_t,u_1,\ldots,u_{t-1})$ of the data).

Under the given conditions, the conditional covariance and coefficients of $F_t(y_1,\ldots,y_t,u_1,\ldots,u_{t-1})$ are independent of both the design $\gamma$ and the data. This is a very important result as it means that in a Gaussian system, recursive filtering calculations may be performed before the system is started and even before the design criteria are known. We have that $F_t(y_1,\ldots,y_t,u_1,\ldots,u_{t-1}) = G(C_t, F_t(y_1,\ldots,y_t,u_1,\ldots,u_{t-1}))$, where $F_t$ is the conditional distribution of $x_{t-n}$ given $(y_1,\ldots,y_t,u_1,\ldots,u_{t-1})$ and the covariances $C_t$ and coefficients in the mean equation $F_t$ are previously computed.
Note that the result in assertion 6 does not hold if we exclude the past controls from the data. In other words, it is not generally true that the conditional distribution of $x_{t-n}$ given $(y_{t,k}, \Theta)$ or $\bigcap_{k=1}^{K} y_{t,k}, \Theta$ and the unconditional distribution of $x_{t-n}$ are Gaussian. On the other hand, if $\gamma$ is affine, then these distributions are Gaussian. However, their means and covariances are not independent of the control laws.

**Linear Control of Linear Gaussian Systems with Arbitrary Pattern**

We now extend our discussion to linear Gaussian systems with arbitrary information patterns. We assume that a design criterion is given and we restrict to linear controls. In this case, the states are affine functions of the primitive random variables (i.e. the noise). Therefore the unconditional distribution of $x_t$ is Gaussian, and hence fully characterized by its mean and covariance. If we define a new system with this mean and covariance as the state vector, then the resulting optimization problem is deterministic. The state equations of the new system can be determined by the properties of a Gaussian system. The system remains finite dimensional but is not linear in general.

Note that due to the non-linearity of the new, deterministic system, there is no guarantee that it can be easily optimized. If we restrict to a strictly classical information pattern and quadratic design criterion, we find that the cost function of the new system is convex over the (affine) control variables. It follows that the optimal affine control is the optimal control for the system. This result does not hold in general.

**Linear Gaussian Systems with Classical Pattern**

Above we claimed that there exists a separation structure for strictly classical systems. In other words, we may restrict the design to the form $\gamma_t = \phi_t \circ F_t$. We also claimed that linear Gaussian systems can be filtered using $F_t(y_1, \ldots, y_t, u_1, \ldots, u_{t-1}) = G(C_t, F_t(y_1, \ldots, y_t, u_1, \ldots, u_{t-1}))$, where $F_t$ is the conditional mean as defined above. We now apply these results to a linear Gaussian system with classical pattern where no assumptions are made about the design criterion and no restrictions
placed on the form of the control laws. We claim that without loss, we may restrict the control law to be of the form $\gamma_t = \phi_t \circ \tilde{F}_t$. Here $\tilde{F}_t$ is affine and characterizes the filter and $\phi_t$ is determined by backward induction and is not affine in general. It follows that $\gamma_t$ is also not affine.

**Linear Quadratic Systems with Classical Pattern**

We now consider a linear system with strictly classical pattern and quadratic criteria. We do not make assumptions about the control, and we do not require that the system be Gaussian. Since the primitive random variables in this system are not Gaussian, the conditional distribution $F_t$ will not be Gaussian in general, and the mean $\tilde{F}_t$ will not be affine in the data in general.

We now construct a deterministic system by fixing the primitive random variables at their mean values and letting $\phi_t^* (x_{t-1})$ be the feedback form of the optimal control for the system at time $t$. Here $\phi_t^*$ must be affine.

**Assertion 7:** In the deterministic system just described, the optimal design is

$\gamma_t (y_1, \ldots, y_t, u_1, \ldots u_{t-1}) = \phi_t^* (\tilde{F}_t (y_1, \ldots, y_t, u_1, \ldots u_{t-1})).$

$\gamma_t$ is not affine in general.

Note that since $\phi_t^*$ is affine, we have

$\gamma_t (y_1, \ldots, y_t, u_1, \ldots u_{t-1}) = E[\phi_t^* (x_{t-1}) | y_1, \ldots, y_t, u_1, \ldots u_{t-1}]$

almost surely. Therefore the optimal control of the stochastic system is the conditional mean of the deterministic system. This is what is known as certainty equivalence.
Classical Linear Quadratic Gaussian Systems
A very common situation is that of a linear Gaussian system with strictly classical pattern and quadratic design criterion where the control is unconstrained. We may apply a number of the above results to find a separation result for this type of system. We find that the optimal control law $\gamma_t$ has the form $\gamma_t = \phi^*_t \circ F_t$ where $\phi^*_t$ and $F_t$ are affine, so $\gamma_t$ is also affine. Here the deterministic control problem (i.e. computing $\phi^*_t$) and the filtering problem (i.e. computing $F_t$) are independent from one another, so $\phi^*_t$ and $F_t$ are computed independently. On the other hand, the deterministic control problem and filtering problem have dual structure because they both depend on the matrix Riccati equations. Note that this result is only valid in the very special case where the noise is Gaussian and the design criterion is quadratic. However, this case is quite common, so the result is a useful one.

Separation for Delayed Sharing Patterns
Consider a general system with $n$-step delayed sharing pattern, and denote by $\delta_t$ the information known to all $k$ stations at time $t$ and denote by $\lambda^k_t$ the information known only to station $k$ at time $t$.

Assertion 8: Without loss, we can restrict our control design to the form

$$\gamma^k_t(\delta_t, \lambda^k_t) = \phi^k_t(\lambda^k_t, F_t(\delta_t))$$

Be aware that though Witsenhausen claims this assumption true for all $n$, Varaiya and Walrand later prove that this assertion is only true for $n = 1$ [2]. For $n > 1$, $F_t(\delta_t)$ consists of all the information about $x_{t-n}$ contained in $\delta_t$ but $\delta_t$ also contains information about the controls selected between $t-n$ and $t$ and this latter information may not be captured by $F_t(\delta_t)$. Therefore this assertion is false for $n > 1$. 
Discussion

In class we discussed separation theory within the topic of partially observed Markov Chain’s. A separation structure was introduced with regards to the linear quadratic Gaussian case and Kalman filtering. This paper brings attention to these results, and further issues in the area of separation of estimation and control.

This paper is a classic paper in the study of estimation and control in that it combines some important initial work done in the area, which has since laid ground work for later research in the study of separation theory.

This paper contains no mathematical theorems instead results are stated as assertions. To turn such assertions into theorems proofs are required, which at the time of this paper had remained an unfinished task. Furthermore, some of the concepts presented in the paper may have been helpful in formulating such proofs. A downside in presenting only assertions and not actual theorems is that the assertions might not actually be correct. Assertion 8 in this paper is false as later discussed in the paper by Pravin Varaiya, and Jean Walard titled On Delayed Sharing Patterns. This result was discussed earlier in this report.

Furthermore, the usefulness in separation theory streams from the advantage it provides by allowing many of the calculations to be done before the system starts, and before the criteria is known.
References
