1. Let $X = (X_1, X_2)^t$ be a 2-dimensional random vector which is uniformly distributed on the unit circle $\{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$. Considering the mean square distortion, find a 2-level Lloyd-Max vector quantizer for $X$. Repeat this for codebooks of size 3 and 4. Calculate the distortion of at least one of these quantizers.

2. Let $Q$ be an $N$-point vector quantizer with codebook $C$ which is optimal in the mean-square sense for the input random vector $X$. Let $Y = TX$, where $T$ is a linear orthogonal transform. Prove that $TC = \{Tc : c \in C\}$ is the codebook of an $N$-point optimal vector quantizer for $Y$.

3. Let $X = (X_1, X_2)$ be a two-dimensional random vector whose probability density function $f_X(x)$ is zero everywhere except for the four shaded sub-squares of the unit square where it has the constant value $A$ (see figure on the next page). In this problem the squared error distortion is considered and we assume throughout that Gersho’s conjecture holds.

   (a) Determine the constant $A$ and compute the marginal densities $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$.

   (b) Let $Q_{1*}$ be an optimal fixed-rate scalar quantizer for $X_1$ at some high rate $R$ (bits/sample) and let $\hat{Q}$ be a two-dimensional fixed-rate product vector quantizer defined by $\hat{Q}(x_1, x_2) = (Q_{1*}(x_1), Q_{1*}(x_2))$. Use the high resolution formula for the distortion of optimal fixed-rate scalar quantizers to calculate distortion of $\hat{Q}$.

   (c) Let $Q^*$ be an optimal fixed-rate two-dimensional vector quantizer for $X$ at some high rate $R$ (bits/sample). Use the high-resolution formula for the distortion of optimal fixed-rate vector quantizers to calculate the distortion of $Q^*$ (recall that $C_2^* = \frac{5}{36\sqrt{3}}$).

   (d) Assuming that $Q^*$ and $\hat{Q}$ have the same high rate $R$ compute the loss (in dB) if $\hat{Q}$ is used to quantize $X$ instead of $Q^*$.
4. (For MATH 877 students.) Let \( \mathbf{X} = (X_1, \ldots, X_k) \) be a \( k \)-dimensional random vector such that each \( X_i \) has zero mean and finite variance. Let \( \mathcal{C} = \{y_1, y_2\} \) be the codebook of a \( k \)-dimensional vector quantizer \( Q \) with smallest possible mean squared distortion among all vector quantizers with two codevectors.

(a) Show that the line segment connecting \( y_1 \) and \( y_2 \) passes through the origin. (Hint: use one of the Lloyd-Max conditions to express the mean of \( \mathbf{X} \).)

(b) Use part (a) to argue that there exists an orthogonal transform \( \mathbf{T} \) on \( \mathbb{R}^k \) such that

\[
\mathbf{T}y_2 = (\|y_2\|, 0, 0, \ldots, 0)^t \quad \text{and} \quad \hat{y}_1 = \mathbf{T}y_1 = (-\|y_1\|, 0, 0, \ldots, 0)^t.
\]

(c) Assume that \( (X_1, \ldots, X_k) \) are independent with each \( X_i \) being zero-mean Gaussian with variance \( \sigma^2 \). Let \( \hat{y}_1 = \mathbf{T}y_1 \), \( \hat{y}_2 = \mathbf{T}y_2 \), and prove that the nearest-neighbor quantizer with codebook \( \{\hat{y}_1, \hat{y}_2\} \) is also MSE-optimal for \( \mathbf{X} \). (Hint: Use Problem 2 and the fact that that \( \mathbf{T}\mathbf{X} \) has the same distribution as \( \mathbf{X} \) since the \( X_i \)'s are i.i.d. Gaussian.)

(d) Recall the unique optimal 2-level scalar quantizer for a Gaussian random variable on pages 24-25 of the Scalar Quantization lecture slides. Use part (c) to find an MSE-optimal two-level vector quantizer for \( \mathbf{X} \). Calculate the distortion of this quantizer.