Suppose \(X_1, X_2, \ldots, X_n\) have the same marginal distribution. Sample-by-sample scalar quantization yields average MSE

\[
E \left[ \frac{1}{n} \sum_{i=1}^{n} (X_i - Q(X_i))^2 \right] = \frac{1}{n} \sum_{i=1}^{n} E[(X_i - Q(X_i))^2] = E[(X_1 - Q(X_1))^2]
\]

Distortion stays the same whether or not the \(X_i\) have statistical dependence \(\Rightarrow\) Cannot exploit statistical dependence between the samples.

Idea: Predict linearly from the previous \(m\) samples

\[
\tilde{X}_n = \sum_{i=1}^{m} a_i X_{n-i}
\]

and quantize the difference signal

\[
e_n = X_n - \tilde{X}_n
\]

Heuristics: Since \(e_n\) is less "spread out" than \(X_n\), it is easier to quantize.

Proposed scheme:

- Heuristics: If the quantization error is small, then \(e_n \approx \hat{e}_n\) and so \(\hat{\tilde{X}}_n \approx X_n\).
- Unfortunately the quantization errors accumulate in this structure.
Let $h_n$ denote the impulse response of the transfer function $\frac{1}{1-P(z)}$.

\[ \hat{X}_n = \hat{e}_n * h_n, \quad X_n = e_n * h_n, \quad \text{and} \quad X_n - \hat{X}_n = (e_n - \hat{e}_n) * h_n \]

Hence

\[ E[(X_n - \hat{X}_n)^2] = E[((e_n - \hat{e}_n) * h_n)^2] \]

\[ = E\left[\sum_{i=0}^{\infty} \frac{(e_{n-i} - \hat{e}_{n-i})}{e_{n-i} - Q(e_{n-i})} h_i \right]^2 \]

Distortion at time $n$ depends on all past quantization errors $e_i - Q(e_i)$

$\Rightarrow$ errors accumulate!

Need a different structure.

**Closed-loop prediction**

Idea: let signal $U_n$ be the prediction of $X_n$ from the past reconstructions

\[ U_n = \sum_{i=1}^{m} a_i \hat{X}_{n-i} \]

\[ X_n \xrightarrow{+} e_n \xrightarrow{Q} \hat{e}_n \xrightarrow{+} \hat{X}_n \]

\[ U_n \xrightarrow{P} \]

**Key feature:** $U_n$ can be generated at both the encoder and the decoder since it depends only on the past reconstruction values.

**Difference quantization**

\[ X_n \xrightarrow{+} e_n \xrightarrow{Q} \hat{e}_n \xrightarrow{+} \hat{X}_n \]

\[ U_n \]

$U_n$ is an arbitrary random variable with finite second moment. Since $e_n = X_n - U_n$ and $\hat{e}_n = \hat{X}_n - U_n$,

\[ E[(X_n - \hat{X}_n)^2] = E[(X_n - U_n - (\hat{X}_n - U_n))^2] \]

\[ = E[(e_n - \hat{e}_n)^2] \]

\[ = E[(e_n - Q(e_n))^2] \]

MSE of reconstruction = MSE of quantizer!

**Equivalent encoder-decoder structure**

\[ X_n \xrightarrow{+} e_n \xrightarrow{Q} i_n \xrightarrow{Q^{-1}} \hat{e}_n \xrightarrow{+} \hat{X}_n \]

\[ U_n \]

- $Q$ outputs both $Q(x) = y_i$ and the index $i$; $Q^{-1}(i) = y_i$.
- By the difference quantization principle

\[ E[(X_n - \hat{X}_n)^2] = E[(e_n - \hat{e}_n)^2] = E[(e_n - Q(e_n))^2] \]

(quantization errors do not accumulate).

- Differential Pulse Code Modulation (DPCM)
How are the predictor coefficients $a_i$ chosen?

- One can attempt to minimize the *closed-loop* MS prediction error:
  \[ E(e_n^2) = E \left[ \left( X_n - \sum_{i=1}^{m} a_i \hat{X}_{n-i} \right)^2 \right] \]

Very hard, since the statistics of the sequence $\{\hat{X}_n\}$ depends on the $a_i$ in a highly nonlinear fashion.

- Instead, choose the $a_i$ to minimize the *open-loop* MS prediction error
  \[ E \left[ \left( X_n - \sum_{i=1}^{m} a_i X_{n-i} \right)^2 \right] \]

Since
\[
E \left[ \left( X_n - \sum_{i=1}^{m} a_i X_{n-i} \right)^2 \right] = E(X_n^2) + \sum_{i=1}^{m} \sum_{k=1}^{m} a_i a_k E(X_{n-i} X_{n-k}) - 2 \sum_{i=1}^{m} a_i E(X_n X_{n-i})
\]

we have
\[
\frac{\partial g}{\partial a_j} = 2 \sum_{k=1}^{m} a_k E(X_{n-j} X_{n-k}) - 2E(X_n X_{n-j})
\]

Setting $\frac{\partial g}{\partial a_j} = 0$ for all $j$ gives $m$ linear equations
\[
\sum_{k=1}^{m} a_k E(X_{n-j} X_{n-k}) = E(X_n X_{n-j}) \quad j = 1, \ldots, m \quad (*)
\]

(It can be shown that the $a_k$ solving $(*)$ indeed minimize $g$.)
Remark The autocorrelation matrix $R$ is symmetric: $r_{ij} = r_{ji}$. It is also nonnegative definite, i.e.,

$$x^t Rx \geq 0 \text{ for all } x = (x_1, \ldots, x_m)^t$$

since

$$x^t Rx = \sum_{i=1}^{m} \sum_{j=1}^{m} x_i E(X_{n-i}X_{n-j}) x_j$$

$$= E \left( \sum_{i=1}^{m} \sum_{j=1}^{m} x_i x_j X_{n-i}X_{n-j} \right)$$

$$= E \left( \sum_{k=1}^{m} x_k X_{n-k} \right)^2 \geq 0$$

Orthogonality principle

Definition The random variables $U$ and $V$ are said to be orthogonal if $E(UV) = 0$.

Prediction error:

$$e_n = X_n - \sum_{k=1}^{m} a_k X_{n-k}$$

We have

$$E(e_n X_{n-j}) = E \left( X_n - \sum_{k=1}^{m} a_k X_{n-k} \right) X_{n-j}$$

$$= E(X_n X_{n-j}) - \sum_{k=1}^{m} a_k E(X_{n-k}X_{n-j})$$

From (*), $a_1, \ldots, a_m$ is optimal if and only if $E(e_n X_{n-j}) = 0$, $j = 1, \ldots, m$.

We have obtained

**Theorem 1 (Orthogonality principle)**

The linear predictor $\hat{X}_n = \sum_{k=1}^{m} a_k X_{n-k}$ is optimal in the MSE sense if and only if the prediction error is orthogonal to all $X_{n-j}$, i.e.,

$$E[(X_n - \hat{X}_n)X_{n-j}] = 0, \quad j = 1, \ldots, m$$

Let $X \perp Y$ denote $E(XY) = 0$ and let $\|X\| \triangleq \sqrt{E(X^2)}$. Then the orthogonality principle states that $a = (a_1, \ldots, a_m)^t$ minimizes

$$\|X_n - \sum_{k=1}^{m} a_k X_{n-k}\|^2$$

if and only if $\hat{X}_n = \sum_{k=1}^{m} a_k X_{n-k}$ satisfies

$$\hat{X}_n - X_n \perp X_{n-j}, \quad j = 1, \ldots, m$$

Illustration: $m = 2$

Let $S = \text{Span}(X_{n-1}, X_{n-2})$

$e_n = X_n - \hat{X}_n$

$\hat{X}_n = \text{arg min}_{Y \in S} \|X_n - Y\| \iff X_n - \hat{X}_n \perp S \iff X_n - \hat{X}_n \perp X_{n-1} \text{ and } X_n - \hat{X}_n \perp X_{n-2}$. 
Wide sense stationary processes

Definition A sequence of r.v.’s \( X_1, X_2, \ldots, X_n, \ldots \) with finite variance is wide sense stationary (WSS) if

(a) \( E(X_k) = \mu \) (constant) for all \( k \) (we assume \( \mu = 0 \))
(b) \( E(X_kX_j) = E(X_{k+i}X_{j+i}) \) for all \( i,j,k \).

Note: From (b), \( E(X_kX_j) \) depends only on the difference \( k - j \). Define

\[
r_j = E(X_nX_{n-j})
\]

and note that \( r_j = r_{-j} \). This is the autocorrelation function of the process.

We assume that \( r_0 = E(X_n^2) > 0 \) since otherwise \( X_n = 0 \) with probability 1 for all \( n \).

For wide sense stationary sequences an optimal \( a = (a_1, \ldots, a_m)^t \) must satisfy

\[
R_m a = v_m
\]

where

\[
R_m = \begin{bmatrix}
r_0 & r_1 & \cdots & r_{m-1} \\
r_1 & r_0 & \cdots & r_{m-2} \\
\vdots & \vdots & \ddots & \vdots \\
r_{m-1} & r_{m-2} & \cdots & r_0
\end{bmatrix}
\]

\[
v_m = \begin{bmatrix}
r_1 \\
r_2 \\
\vdots \\
r_m
\end{bmatrix}
\]

If \( R_m \) is invertible

\[
a = R_m^{-1} v_m
\]

Note: \( R_m \) and the optimal \( a \) depend only on the prediction order \( m \), but not on the time index \( n \).

- Can be shown: for an optimal \( c \) we have \( c^t R_{m+1} c = 0 \) iff \( R_{m+1} \) is singular.
- Hence \( E(c_n^2) = 0 \) if and only if the autocorrelation matrix \( R_{m+1} \) is singular.
- If \( R_m \) is nonsingular for all \( m \geq 1 \), the WSS process is called nondeterministic.
- Another look at \( E(c_n^2) \): since \( c = (1, -a)^t \)

\[
E(c_n^2) = c^t R_{m+1} c
\]

\[
= r_0 - 2a^t v_m + a^t R_m a
\]

- If \( a \) is optimal, \( v_m = R_m a \), so

\[
E(c_n^2) = r_0 - a^t v_m = r_0 - \sum_{j=1}^m a_j r_j
\]
Theorem 2

For the mth-order optimal prediction error we always have

\[ E(e_n^2) < E(X_n^2) \]

unless \( X_n \) and \( X_{n-j} \) are uncorrelated (i.e., \( E(X_nX_{n-j}) = 0 \)) for all \( j = 1, \ldots, m \).

Proof: Assume \( r_j = E(X_nX_{n-j}) \neq 0 \) for some \( j \in \{1, \ldots, m\} \). Then

\[ E(e_n^2) = \min_{a_1, \ldots, a_m} E \left[ \left( X_n - \sum_{i=1}^{m} a_i X_{n-i} \right)^2 \right] \leq \min_a E \left[ (X_n - aX_{n-j})^2 \right] \]

From the orthogonality principle the optimum \( a \) solves

\[ E[(X_n - aX_{n-j})X_{n-j}] = 0, \]

which gives \( a = r_j/r_0 \). Thus

\[
\begin{align*}
E(e_n^2) & \leq E[(X_n - aX_{n-j})^2] = r_0 + a^2r_0 - 2ar_j \\
& = r_0 - \frac{r_j^2}{r_0} < r_0 = E(X_n^2)
\end{align*}
\]

\[ \square \]

System SNR (not in dB):

\[
\frac{E(X_n^2)}{E[(X_n - \hat{X}_n)^2]} = \frac{E(X_n^2)}{E[(e_n - \hat{e}_n)^2]} = \frac{E(X_n^2)}{E(e_n^2)} \frac{E(e_n^2)}{E[(e_n - \hat{e}_n)^2]} G_Q
\]

Thus

\[ \text{SNR}_{\text{sys}} = 10 \log_{10} G_{\text{clp}} + \text{SNR}_Q \quad \text{[dB]} \]

Note:

- \( G_{\text{clp}} \) and \( G_Q \) are interdependent \( \Rightarrow \) one cannot maximize one while keeping the other fixed.
- Increasing \( G_{\text{clp}} \) does not necessarily increase \( \text{SNR}_{\text{sys}} \) (although in practice it most often does).

Performance gain over scalar quantization

Assume \( X_1, X_2, \ldots, X_n, \ldots \) is WSS

Recall the difference quantization principle:

\[ E[(X_n - \hat{X}_n)^2] = E[(e_n - \hat{e}_n)^2] = E[(e_n - Q(e_n))^2] \]

Quantization errors do not accumulate. Can assess the performance gain over baseline scalar quantization.

Closed-loop prediction gain:

\[ G_{\text{clp}} = \frac{E(X_n^2)}{E(e_n^2)} \]

Intuitively, the larger \( G_{\text{clp}} \), the more compact and easier-to-quantize \( e_n \) is.

Approximations (assuming high-resolution quantization):

(a) \( G_{\text{clp}} \approx G_{\text{olp}} = \frac{E(X_n^2)}{E(\sum_{i=1}^{m} a_i X_{n-i})^2)} G_Q \)

(accurate if the quantization error is small)

(b) \( G_Q \) is mainly determined by the rate of \( Q \) and not the statistics of \( e_n \) (recall the 6 dB/bit rule).

Then

\[ \text{SNR}_{\text{sys}} \approx \text{SNR}_Q + 10 \log_{10} G_{\text{olp}} \]

(a) and (b) \( \Rightarrow \) the overall gain (in dB) over direct scalar quantization is (approximately)

\[ 10 \log_{10} G_{\text{olp}} \]
According to the approximation, to maximize SNR$_{sys}$, one should minimize the MSE open-loop prediction error.

With the optimal coefficients $a_1, \ldots, a_m$, the maximum gain is

$$G_{clp} \approx G_{olp} = \frac{E(X_n^2)}{E[(X_n - \sum_{i=1}^{m} a_i X_{n-i})^2]} = \frac{r_0}{r_0 - \sum_{i=1}^{m} a_i r_i}$$

Note: This analysis is not precise and has limited accuracy. Rigorous analysis is hard.

**Computing the prediction coefficients**

Assuming $R_m$ is invertible, the optimal coefficient vector is

$$a = R_m^{-1} v_m$$

- Computing $R_m$ involves inverting an $m \times m$ matrix $\Rightarrow O(m^3)$ computations (Gaussian elimination)
- If $R_m$ is ill-conditioned, numerical inversion is even harder
- $R_m$ has structure: symmetric and positive definite.
- $R_m$ is also a Toeplitz matrix: it has constant negative-sloping diagonals.
- The structure of $R_m$ can be exploited to solve the problem using $O(m^2)$ computations using the Levinson-Durbin algorithm.

**Design of predictors from empirical data**

- In practice the autocorrelation values $r_0, r_1, \ldots, r_m$ may not be known or may change with time (i.e., the WSS model is not accurate)
- Want to use observed source samples $x_0, x_1, \ldots, x_M$ to design the $m$th order predictor ($M \gg m$).
- Design philosophy: minimize the empirical prediction error, i.e., the time average of

$$\left(x_n - \sum_{i=1}^{m} a_i x_{n-i}\right)^2$$

**Adaptive DPCM (ADPCM)**

- For DPCM the predictor coefficients are time-invariant (recall the WSS assumption). In practice, the source is often non-stationary and the coefficients need readjustment to match the source’s local behavior.
- Adaptive prediction: The prediction coefficients are periodically recalculated and sent to the decoder as side information.
- Quantizer adaptation: The quantizer is adaptively updated to match the source’s local statistics (e.g. the step-size of uniform quantizer).
- Widely used in speech and video coding.
- Typical bit rates for speech: 32, 24, and 16 kbits/s. Compare to standard telephone quality speech at 64 kbits/s (8000 samples/s quantized at 8 bits/sample).
**Analysis-synthesis schemes**

- Construct a parametric *predictive* model of how the source generates the signal.
- Extract the model parameters, and transmit them.
- At the receiver side use the parameters to synthesize an approximation to the source signal.
- Most successful applications in speech coding because *good models* for speech generation exist.
- Intelligible speech quality at 2.4 kbits/s (0.3 bit/sample), good quality at 16 kbits/s (2 bits/sample).
- Applications in mobile telephony, VoIP, etc.