Queen's University - Math 844 **Fall 2022**

Problem Set #1

Posted: Thursday, 22/9/2022
Due: Thursday, 29/9/2022

Note: In all that follows, the space $\mathbb{R}^n$ is endowed with its usual topology.

1. Let (X,τ) be a topological space. Consider the product space $X\times X$ with the product topology. Let $\Delta=\{(x,y)\in X\times X|x=y\}$ (the "diagonal" set). Show that X is Hausdorff $\Leftrightarrow$ Δ is closed in $X\times X$.
2. Let $f:\mathbb{R}^n\to\mathbb{R}$ continuous. Show that if $n\geq 2$, then f cannot be injective.
3. Let (X,τ) be a topological space, and consider $\{0,1\}$ with the discrete topology τ_d. Show that (X,τ) is a connected topological space iff $\forall f:(X,\tau)\to(\{0,1\},\tau_d)$ we have: f continuous $\Rightarrow$ f constant. Use this to show that $\forall A\subset X$, A connected $\Rightarrow$ $\bar{A}$ connected.
4. We defined the circle S^1 as a topological space in two ways:
   (a) As the quotient of $\mathbb{R}$ by the equivalence relation $\sim$ (with $x\sim y \Leftrightarrow x-y\in\mathbb{Z}$) with the quotient topology, and
   (b) as the subset $\{(x,y)\in\mathbb{R}^2|x^2+y^2=1\}$ of $\mathbb{R}^2$ with the subspace topology induced by the inclusion $S^1\hookrightarrow\mathbb{R}^2$.

   Show that the topological spaces obtained in (a) and (b) are homeomorphic.
5. Consider the subset X of $\mathbb{R}^2$ given by $X=\{(x,y)\in\mathbb{R}^2|xy=0\}$, endowed with the subspace topology τ induced by inclusion in $\mathbb{R}^2$. Show that there is no open neighborhood of $(0,0)$ in (X,τ) homeomorphic to an open subset of $\mathbb{R}^n$ (for any $n\in\mathbb{N}$). (As a result, (X,τ) cannot admit any C^0 manifold structure – hence no C^k manifold structure for $k\in\mathbb{N}\cup\{\infty\}$).
6. Let (X,τ) be a topological space, and let $\{\mathcal{A}_\alpha\}_\alpha$ be the (possibly empty) set of all C^∞ atlases on (X,τ). Show that the relation of C^∞ compatibility between these atlases is an equivalence relation (each equivalence class being a distinct C^∞ structure on (X,τ)).
7. $\forall n\in\mathbb{N}$, define the unit sphere $S^n=\{\mathbf{x}\in\mathbb{R}^{n+1}|\|\mathbf{x}\|=1\}$ of $\mathbb{R}^{n+1}$, and consider it as a topological subspace of $\mathbb{R}^{n+1}$.
   (a) Show that S^n is a compact path-connected topological space (use the fact that a subset of $\mathbb{R}^k$ is compact iff it is closed and bounded).
   (b) Construct a C^∞ atlas on S^n using stereographic projection (as we did in class for S^1), and show that it is indeed a C^∞ atlas.
8. Let $p,n\in\mathbb{N}^\star$ with $p\leq n$, and let $\mathcal{M}_{n,p}$ be the vector space of all real $n\times p$ matrices (with its canonical topology as a finite-dimensional vector space over $\mathbb{R}$), and let $\hat{\mathcal{M}}_{n,p}$ be the subset of $\mathcal{M}_{n,p}$ consisting of matrices of rank p.
   (a) Show that $\hat{\mathcal{M}}_{n,p}$ is an open subset of $\mathcal{M}_{n,p}$. We shall consider $\hat{\mathcal{M}}_{n,p}$ as a topological subspace of $\mathcal{M}_{n,p}$.
   (b) Consider the following relation on $\hat{\mathcal{M}}_{n,p}$: $M_1\sim M_2 \Leftrightarrow \exists A\in GL(p,\mathbb{R}): M_1=M_2A$. Show that $\sim$ is an equivalence relation on $\hat{\mathcal{M}}_{n,p}$. We denote the quotient space $\hat{\mathcal{M}}_{n,p}/\sim$ equipped with the quotient topology by $\mathcal{G}_{n,p}$.
   (c) We now endow $\mathcal{G}_{n,p}$ with a C^∞ structure. For each ordered $p-$tuple $J=(i_1,\cdots,i_p)$ of distinct integers $1\leq i_1<i_2<\cdots<i_p\leq n$, and for each $M\in\mathcal{M}_{n,p}$, we denote by M_J the $p\times p$ matrix obtained from M by keeping only rows $i_1,i_2,\cdots,i_p$. We similarly denote by J' the ordered $(n-p)-$tuple of integers in $\{1,\cdots,n\}$ complementary to those of J, and by $M_{J'}$ the $(n-p)\times p$ submatrix of M obtained by keeping only the rows with indices in J'. For each such $p-$tuple J (note that there are $\frac{n!}{(n-p)!p!}$ such J's in total), we define $V_J=\{[M]\in\mathcal{G}_{n,p}|\det(M_J)\neq 0\}$. Show that V_J is well-defined, and is open in $\mathcal{G}_{n,p}$.
   (d) For each $p-$tuple J as in (c), we define the mapping $\phi_J:V_J\to\mathbb{R}^{p(n-p)}$, where $\forall[M]\in V_J$, $\phi_J([M])$ is the $p(n-p)-$tuple of entries (with respect to some defined order) of the matrix $(M_{J'})(M_J)^{-1}$. Show that for each J (as in (c)) the map ϕ_J is well-defined and is a homeomorphism.
   (e) Show that the family $\{(V_J,\phi_J)\}_J$ defines a C^∞ atlas, and hence a C^∞ structure, on $\mathcal{G}_{n,p}$.

   Note:
   (i) $\mathcal{G}_{n,p}$ is called the real Grassmannian manifold of $p-$dimensional subspaces in $\mathbb{R}^n$.
   (ii) $\mathcal{G}_{n,1}$ is nothing other than $\mathbb{RP}^n$.