Queen's University - Math 844 **Fall 2022**

Problem Set #4

Posted: Friday, 4/11/2022
Due: Friday, 11/11/2022

1. Let M be a C^∞ $n-$manifold, $(U, \phi = (x^1, \cdots, x^n))$ and $(U, \psi = (y^1, \cdots, y^n))$ two local charts on M (with same domain U), and X a C^∞ vector field on M. We can write $X|_U = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i} = \sum_{i=1}^n b^i \frac{\partial}{\partial y^i}$, where the a^i and b^i are uniquely defined C^∞ functions on U; find the relation between the a^i and the b^i.

2. Let M be a C^∞ manifold of dimension 2, and let X_1, X_2 be C^∞ vector fields on M. Let $p \in M$, and let $(U, \phi = (x^1, x^2))$ be a local coordinate chart of M around p. Assume $X_1 = \frac{\partial}{\partial x_1}$, $X_2 = x_1^l \frac{\partial}{\partial x_2}$ in U, where $l \in \mathbb{N}$. Let $k \in \mathbb{N}$ with $k \neq l$; show that there exists no local chart $(V, \psi = (y^1, y^2))$ of M around p such that $X_1 = \frac{\partial}{\partial y_1}$, $X_2 = y_1^k \frac{\partial}{\partial y_2}$ in V.

3. Consider the following C^∞ vector fields on $\mathbb{R}^3$ (with the global canonical coordinate chart $(\mathbb{R}^3, \phi = (x, y, z))$: $X_1 = x\frac{\partial}{\partial y}$, $X_2 = y\frac{\partial}{\partial z}$, $X_3 = z\frac{\partial}{\partial x}$, $Y_1 = x^2\frac{\partial}{\partial y}$, $Y_2 = y^2\frac{\partial}{\partial z}$, $Y_3 = z^2\frac{\partial}{\partial x}$. Show that there is no local diffeomorphism f of $\mathbb{R}^3$ for which $Y_k = f_\star(X_k)$ $\forall k \in \{1, 2, 3\}$.

4. Let G be a Lie group (with identity element e), M a C^∞ manifold. A (smooth, left) action of G on M is a C^∞ mapping $a : G \times M \to M$ such that $a(e, p) = p$ and $a(s, a(t, p)) = a(st, p)$ $\forall p \in M$ and $\forall s, t \in G$. We shall write $s \cdot p$ instead of $a(s, p)$. Let now $p \in M$ and let $S_p = \{s \in G \mid s \cdot p = p\}$ be the stabilizer of p in G. Show that S_p is a Lie subgroup of G.

5. Let G be a Lie group, M a C^∞ manifold, and assume given a (smooth, left) action $(s, p) \mapsto s \cdot p$ of G on M. Consider the relation $\sim$ given on M by $p \sim q \Leftrightarrow \exists s \in G : p = s \cdot q$; it is easy to verify that $\sim$ is an equivalence relation on M. We denote the quotient set $M/\sim$ by M/G instead. Let $\pi : M \to M/G$ denote the quotient map.
   - Show that π is an open mapping.
   - Show that M/G is second countable.
   - Let $R = \{(p, q) \in M \times M \mid p \sim q\}$; show that M/G is Hausdorff if and only if R is closed in $M \times M$.

6. Let G be a Lie group of dimension n; show that the tangent bundle TG of G is *trivial*, i.e. there exists a diffeomorphism $\phi : TG \to G \times \mathbb{R}^n$ such that $pr_1 \circ \phi = \pi$, where $pr_1 : G \times \mathbb{R}^n \to G$ is the projection on the first factor and $\pi : TG \to G$ is the canonical surjection, and such that $\forall g \in G$, the mapping $\phi|_{\pi^{-1}(g)} : \pi^{-1}(g) \to \{g\} \times \mathbb{R}^n$ is a vector space isomorphism. (NOTE: We will prove later that TS^2 is not trivial; this implies that S^2 cannot admit a Lie group structure, i.e. a group structure compatible with its smooth structure).

7. Let $n \in \mathbb{N}$, $n \geq 2$, and consider the canonical inclusion $i : S^{n-1} \hookrightarrow \mathbb{R}^n$. $\forall p \in S^{n-1}$, we define the vector subspace $p^\perp$ of $\mathbb{R}^n$ by:
$$p^\perp = \{v \in \mathbb{R}^n \mid \langle p, v\rangle = 0\},$$
where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$.
   (a) Define a canonical vector space isomorphism between $T_p i(T_p S^{n-1})$ and $p^\perp$ for all $p \in S^{n-1}$.
   (b) Let $U \subset \mathbb{R}^n$ open, with $S^{n-1} \subset U$; show that if $v : U \to \mathbb{R}^n$ is a nowhere vanishing C^∞ vector field on U with $v(p) \in p^\perp$ $\forall p \in S^{n-1}$, then there exists a C^∞ vector field w on S^{n-1} with $T_p i(w(p)) = v(p)$ and $w(p) \neq 0$ $\forall p \in S^{n-1}$.
   (c) Let $(e_i)_{i=1}^n$ be the canonical basis of $\mathbb{R}^n$, and let $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a bilinear mapping on $\mathbb{R}^n$ with no zero divisor (i.e. $x, y \neq 0 \Rightarrow B(x, y) \neq 0$). Consider the vector space isomorphism $\alpha_1 : \mathbb{R}^n \to \mathbb{R}^n$ given by $\alpha_1(x) = B(x, e_1)$, $\forall x \in \mathbb{R}^n$, and, $\forall i \in \{1, \cdots, n\}$, consider the vector fields $w_i : \mathbb{R}^n \to \mathbb{R}^n$ on $\mathbb{R}^n$ defined by $w_i(x) = B(\alpha_1^{-1}(x), e_i)$, $\forall x \in \mathbb{R}^n$, and, $\forall i \in \{1, \cdots, n\}$, define the vector fields $v_i : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ on $\mathbb{R}^n \setminus \{0\}$ by:
$$v_i(x) = w_i(x) - \langle w_i(x), \frac{x}{\|x\|}\rangle \frac{x}{\|x\|},\ \forall x \in \mathbb{R}^n \setminus \{0\},$$
where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$. By studying the vector fields v_i, $i = 1, \cdots, n$, deduce that the tangent bundle TS^{n-1} of S^{n-1} must be trivial.
   (d) Deduce from (c) that the tangent bundle TS^1 of S^1 is trivial (hint: identify $\mathbb{R}^2$ with $\mathbb{C}$ and define the desired bilinear form on $\mathbb{R}^2$ through complex multiplication in $\mathbb{C}$).

NOTE: It can be proved in the same way that TS^3 and TS^7 are trivial as well (corresponding to the multiplication of quaternions in $\mathbb{R}^4$ and octonions in $\mathbb{R}^8$, respectively). On the other hand, since TS^2 is not trivial (to be proved later), there can be no "multiplication" operation in $\mathbb{R}^3$ (with no zero divisor) compatible with the group structure of $\mathbb{R}^3$; in particular, the group structure of $\mathbb{R}^3$ cannot be extended to a field structure.