MATH 891 Analysis I Autumn 2017

Assignment 2, Due Oct. 11

1) Let $\{\mu_i\}_{i=1}^{\infty}$ be a sequence of measures on a measurable space $(X, \mathfrak{M})$ and $\{\alpha_i\}_{i=1}^{\infty}$ be a sequence with $\alpha_i \geq 0$. Let $\mu(A) = \sum_{i=1}^{\infty} \alpha_i \mu_i(A)$. Show that μ is a measure.

2) Let X be a set and $\{A_n\}_{n=1}^{\infty}$ be a sequence of subsets of X. Let χ_{A_n} be the indicator function of A_n. Let $\overline{\lim}_n A_n = \{x \in X \mid x \in A_n \text{ for infinitely many } n\text{'s}\}$ and $\underline{\lim}_n A_n = \{x \in X \mid x \in A_n \text{ for all but finitely many } n\text{'s}\}$.

i) Show $\overline{\lim}_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$.

ii) Show $\underline{\lim}_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$.

iii) Show $\limsup_n \chi_{A_n} = \chi_A$ where $A = \overline{\lim}_n A_n$

iv) Show $\liminf_n \chi_{A_n} = \chi_B$ where $B = \underline{\lim}_n A_n$

3) Let $(X, \mathfrak{M})$ be a measurable space and $\mu : X \to [0, \infty]$. We say that μ is *finitely additive* if whenever $E_1, \ldots, E_n \in \mathfrak{M}$ are pairwise disjoint we have $\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i)$.

If we have $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$ for every sequence $\{E_i\}_{i=1}^{\infty}$ (even if they are not pairwise disjoint), we say that μ is *countable subadditive*.

Show that if μ is both finitely additive and countably subadditive then it is countably additive.

In the next two exercises we make use of Lebesgue measure on the σ-algebra of Borel sets. For the context of these two questions we work with $X = [0, 1]$ and $\mathfrak{M}$ is the σ algebra of Borel subsets of $[0, 1]$. We will denote Lebesgue measure by m. The main point we need to know about Lebesgue measure is that if $I \subseteq [0, 1]$ is an interval (open, closed, half-open) then $m(I) = \ell(I)$ where ℓ is the length. Thus $m((1/2, 3/4]) = 1/4$.

4) Let m denote Lebesgue measure on $[0, 1]$.

i) Show that for every $\epsilon > 0$ there exists an open set $G \subseteq [0, 1]$ such that $m(G) < \epsilon$ and $\mathbb{Q} \cap [0, 1] \subseteq G$.

ii) Show that there is a closed set $C \subseteq [0, 1]$ such that $m(C) > 0$ but C does not contain any open interval.

5) Let m denote Lebesgue measure on $[0, 1]$, and let $\{A_n\}_n$ be a sequence of measurable subsets of $[0, 1]$. Suppose that there is $\epsilon > 0$ such that for all n, $m(A_n) > \epsilon$. Show that there is at least one $x \in [0, 1]$ such that x is in infinitely many A_n's.