MATH 891
Analysis I
Autumn 2017

Assignment 5, Due Dec. 4

1) Show that in a finite dimensional normed linear space X all linear maps from X to $\mathbb{C}$ are bounded.

2) Recall that given a normed linear space $(X, \|\cdot\|)$, we let $X^* = \{\phi : X \to \mathbb{C} \mid \phi$ is bounded and linear$\}$. X^* is called the *dual* space of X.
Let $\{x_n\}_n \subseteq X$ be a sequence in X. We say that $\{x_n\}_n$ converges *strongly* to $x \in X$ if $\lim_n \|x_n - x\| = 0$. We say that $\{x_n\}_n$ converges *weakly* to $x \in X$ if for every $\phi \in X^*$ we have $\lim_n \phi(x_n) = \phi(x)$.

i) Show that if $\{x_n\}_n$ converges weakly to x then $\|x\| \leq \limsup \|x_n\|$.(1)

ii) Suppose that X is a Hilbert space. Show that if $\{x_n\}_n$ converges weakly to x and $\lim_n \|x_n\| = \|x\|$, then $\{x_n\}_n$ converges strongly to x.

3) For $f \in L^\infty[0,1]$, define $M_f : L^2[0,1] \to L^2[0,1]$ by $M_f(g)(x) = f(x)g(x)$. Show that M_f is a bounded linear operator and $\|M_f\| = \|f\|_\infty$.

(1) You may use Theorem 5.20 in Rudin if you wish. It states that for every $x \neq 0$ in a normed linear space X, there is $\phi \in X^*$ with $\|\phi\| = 1$ and $\phi(x) = \|x\|$.

4) Show that for every irrational number α and every continuous function f on $\mathbb{R}$ with period 1

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) = \int_0^1 f(t)dt$$

(*Hint:* Check this first for $f(x) = e^{2\pi i k x}$, and then use the density of the trigonometric polynomials.)