Math 891: Core Course in Analysis I, Fall 2017

Course Outline

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The course follows the first eight chapters of Walter Rudin, *Real and Complex Analysis*, 3rd. ed. Some sections will only be discussed in summary form. I will indicate which as the term unfolds. I will follow the construction from *Real Analysis*, 2^{nd} edition by Gerald Folland.

There will be five assignments to submit (40 %), an in-class presentation on some material from Rudin (30 %), and a final examination (30 %). In order to pass the course you must receive a grade of at least 50% on the final exam.

1) Rudin Ch. 1: σ-algebras, measures on σ-algebras, integrals, null sets, convergence theorems

2) The construction of Lebesgue measure, Borel and Stieljtes measures (partly from Rudin Ch. 2, partly from Folland §1.4 and §1.5)

3) Rudin Ch. 8: product measures and Fubini's theorem (§1, 2, and 3).

4) Rudin Ch. 3: L^p spaces, the inequalities of Jensen, Hölder, and Minkowski

5) Rudin Ch. 4: Hilbert spaces, subspaces and orthonormal bases

6) Rudin Ch. 5: Banach space techniques: Baire category theorem, Hahn-Banach theorem, Principle of Uniform Boundedness, Open Mapping theorem

7) Rudin Ch. 6: complex measures and the Radon-Nikodym theorem

8) Rudin Ch. 7: the fundamental theorem of calculus for Lebesgue theory

IN THE SEVENTEENTH CENTURY it was discovered that one could find the area under the graph of a function by first finding an anti-derivative for the function and then use the fundamental theorem of calculus. The curves in question were usually assumed to be piecewise smooth, but the limits of the theory were not carefully probed until Fourier's theory of trigonometric series early in the nineteenth century. By the middle of the nineteenth century a satisfactory theory of integration had been established for continuous functions using what we call the Riemann integral.

By the end of the nineteenth century mathematicians were asking themselves for which curves can one define a length, for which regions of the plane can one define an area, for which regions of space can one define a volume? Such sets came to called *measurable*. This idea led to a new kind of integral called the *Lebesgue integral*, which agrees with Riemann's for piecewise continuous functions but was much more general.

Lebesgue's theory was much better suited to unbounded functions and unbounded regions. Moreover there are theorems which make it easy to interchange a limit with an integral sign. For example, in Fourier analysis an important integral is

$$\int_0^\infty \frac{\sin x}{x}\,dx.$$

Here is a way of evaluating this integral which uses methods common in the nineteenth century.

$$\begin{aligned}
\int_0^\infty \frac{\sin x}{x}\,dx &= \int_0^\infty \lim_{a\to 0^+} e^{-ax}\frac{\sin x}{x}\,dx = \lim_{a\to 0^+}\int_0^\infty e^{-ax}\frac{\sin x}{x}\,dx \\
&= \lim_{a\to 0^+}\int_0^\infty e^{-ax}\int_0^1 \cos(xy)\,dy\,dx = \lim_{a\to 0^+}\int_0^1\int_0^\infty e^{-ax}\cos(xy)\,dx\,dy \\
&= \lim_{a\to 0^+}\int_0^1 \left[e^{-ax}\frac{-a\cos(xy)+y\sin(xy)}{a^2+y^2}\right]_0^\infty dy = \lim_{a\to 0^+}\int_0^1 \frac{a}{a^2+y^2}\,dy \\
&= \lim_{a\to 0^+}\left[\arctan\left(\frac{y}{a}\right)\right]_0^1 = \frac{\pi}{2}
\end{aligned}$$

Quite a few times one must interchange an integral sign with a limit or another integral sign. In Riemann's theory one usually needs uniform convergence to do this, something we don't have here. One now usually computes Fourier integrals by contour integration. Some of the interchanges required are now standard results in measure theory; the remaining ones require ad-hoc methods.

Here is another even more troubling example.

$$\begin{aligned}
\int_{-\infty}^\infty \frac{\tan x}{x}\,dx &= \sum_{k=-\infty}^\infty \int_{(k-1)\pi}^{k\pi} \frac{\tan x}{x}\,dx = \sum_{k=-\infty}^\infty \int_0^\pi \frac{\tan(y+(k-1)\pi)}{y+(k-1)\pi}\,dy \\
&= \sum_{k=-\infty}^\infty \int_0^\pi \frac{\tan y}{y+(k-1)\pi}\,dy = \int_0^\pi \tan y \sum_{k=-\infty}^\infty \frac{1}{y+(k-1)\pi}\,dy \\
&= \int_0^\pi \tan y \left\{\frac{1}{y} + \sum_{k=1}^\infty \frac{1}{y-k\pi} + \frac{1}{y+k\pi}\right\} dy \\
&= \int_0^\pi \tan y \left\{\frac{1}{y} + 2y\sum_{k=1}^\infty \frac{1}{y^2-k^2\pi^2}\right\} dy = \int_0^\pi \tan y \cot y\,dy = \pi.
\end{aligned}$$

Yet another example which comes from Fourier's 1822 memoir concerns a function on **R**. For any $L > 0$ we may take the part of the function between $-L$ and L and expand in a Fourier series:

$$\begin{aligned} f(x) &= \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt\cos\left(\frac{n\pi x}{L}\right) \\ &\quad + \sum_{N=1}^{\infty} f(t)\sin\left(\frac{n\pi t}{L}\right)dt\sin\left(\frac{n\pi x}{L}\right) \\ &= \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \sum_{n=1}^{\infty}\frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}(t-x)\right)dt \end{aligned}$$

If $\int_{-\infty}^{\infty}|f(t)|\,dt < \infty$ then $\frac{1}{2L}\int_{-L}^{L} f(t)\,dt \to 0$ as $L \to \infty$. Fourier then argued that

$$\lim_{L\to\infty}\sum_{n=1}^{\infty}\cos\left(\frac{n\pi}{L}(t-x)\right)\frac{1}{L} = \frac{1}{\pi}\int_{0}^{\infty}\cos(s(t-x))\,ds \tag{1}$$

as we might be tempted to do, regarding the left hand side as a Riemann sum. He then concluded that

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos(s(t-x))\,dt\,ds.$$

Equation 1 provoked quite a bit of controversy. All modern analyses of Fourier's integrals rely on Lebesgue's integral.

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