Math 891: Core Course in Analysis I, Fall 2017

Speaking Topics

Class members shall form themselves into pairs. Each pair shall choose a topic. Topics are available on a first come first served basis. Some topics depend on other topics. A team can use the results of these other topics freely. Each talk should last twenty minutes with a few minutes for questions at the end. The team members should participate equally in preparing the presentation and in speaking to the class. Each team should submit a written summary outlining the presentation at least one day before the presentation. Not all the topics are of same degree of difficulty. The presentations will be graded on: correctness of mathematics, written submission, clarity of presentation, and ability to respond to questions.

1) Let $(\mathbb{R}, \mathfrak{M}, m)$ denote the measure space of the Lebesgue measurable subsets of the real line and m Lebesgue measure. By m^* we denote Lebesgue outer measure.

a) Show that for any $A \subset \mathbb{R}$ there is $E \in \mathfrak{M}$ such that $A \subseteq E$ and $m^*(A) = m(E)$.

b) Let $V \subset \mathbb{R}$ be in $\mathfrak{M}$ with $m(V) < \infty$. Show that for any subset $A \subseteq V$ there is $E \in \mathfrak{M}$ such that $E \subset A$ and $m^*(V \setminus A) = m(V \setminus E)$.

c) Let $V \subset \mathbb{R}$ be in $\mathfrak{M}$ with $m(V) < \infty$. Let $E \subset V$ and suppose that $m(V) = m^*(E) + m^*(V \setminus E)$. Show that $E \in \mathfrak{M}$.

2) Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $\mathfrak{N} = \{E \subset X \mid \exists A, B \in \mathfrak{M}$ such that $A \subseteq E \subseteq B$ and $\mu(B \setminus A) = 0\}$. I showed that $\mathfrak{N}$ is a σ-algebra containing $\mathfrak{M}$.[(1)]

[1] Rudin: Thm. 1.36

a) For $E \in \mathfrak{N}$ and $A, B \in \mathfrak{M}$ such that $A \subseteq E \subseteq B$ and $\mu(B \setminus A) = 0$, let $\nu(E) = \mu(A)$. Show that the value of $\nu(E)$ does not depend on the choice of A and B.

b) Show that ν is a measure on $\mathfrak{N}$.

c) Show that for $A \in \mathfrak{M}$ we have $\mu(A) = \nu(A)$.

3) Let $\mathfrak{B}$ be the σ-algebra of Borel subsets of $\mathbb{R}$ and let μ be a measure on $(\mathbb{R}, \mathfrak{B})$ such that μ is translation invariant ($\mu(x + E) = \mu(E)$ for $x \in \mathbb{R}$ and $E \in \mathfrak{B}$) such that $\mu((0, 1]) = 1$. Show that μ is the restriction of Lebesgue measure to $\mathfrak{B}$. (Use Dudley Theorem 3.1.10 and Rudin 2.20 (*d*).[(2)])

[2] R. M. Dudley, *Real Analysis and Probability*, Cambridge U. Press, 2002

4) Let $x, y \in [0,1)$ and write $x+y$ to mean addition modulo 1 (*i.e.* $\mathbb{R}/\mathbb{Z}$). For $x, y \in [0,1)$ let $x \sim y$ mean that $x-y$ is rational. By the axiom of choice there is $P \subset [0,1)$ that contains exactly one representative from each equivalence class. Let $\{r_i\}_i$ be an enumeration of $\mathbb{Q} \cap [0,1)$ with $r_0 = 0$. Let $P_i = P + r_i$. Show that

a) $[0,1) = \cup_{i=0}^{\infty} P_i$ and $P_i \cap P_j = \emptyset$ for $i \neq j$;

b) P is not Lebesgue measurable;

c) if $E \subset P$ is measurable, then $m(E) = 0$;

d) if $E \subset [0,1)$ is an subset with $m^*(E) > 0$, then E contains a non-measurable set.

5) Let $(X, \mathfrak{M})$ and $(Y, \mathfrak{N})$ be measurable spaces and $\mathcal{P}$ the smallest σ-algebra containing the measurable rectangles $E \times F$ with $E \in \mathfrak{M}$ and $F \in \mathfrak{N}$. Show that $\mathcal{P}$ is the smallest monotone class containing the elementary sets (the finite disjoint unions of measurable rectangles).(3)

[3] Rudin: Thm. 8.3

6) Let μ be a finite measure on X with $\mu(X) < \infty$. Let $\{f_n\}_n$ be a sequence of measurable functions on X and f another measurable function such that for every $\epsilon > 0$ there is N such that for all $n \geq N$, $\mu(\{x \mid |f_n(x) - f(x)| > \epsilon\}) < \epsilon$. Then we say that $\{f_n\}_n$ *converges in measure* to f. Show that

a) if $f_n(x) \to f(x)$ almost everywhere then $\{f_n\}_n$ converges to f in measure;

b) for $1 \leq p \leq \infty$, if $\|f_n - f\|_p \to 0$ then $\{f_n\}_n$ converges to f in measure;

c) if $\{f_n\}_n$ converges in measure to f then $\{f_n\}_n$ has a subsequence that converges to f almost everywhere.

7) Let $\mathcal{I}$ be a collection of intervals (open, closed, or half open, but all with non-empty interior) and $E \subseteq \mathbb{R}$ be a set. We say that $\mathcal{I}$ *covers* E *in the sense of Vitali* if for all $x \in E$ and all $\epsilon > 0$, $\exists I \in \mathcal{I}$ such that $x \in I$ and $l(I) < \epsilon$.

Show that if $m^*(E) < \infty$ and $\mathcal{I}$ covers E in the sense of Vitali, then for all $\epsilon > 0$ there exist $I_1, \ldots, I_n \in \mathcal{I}$ such that

$$m^*\big(E \setminus \bigcup_{i=1}^{n} I_i]\big) < \epsilon.$$

This is called *Vitali's covering lemma*.(4)

[4] H. L. Royden *Real Analysis*, 3rd ed., Lemma 1 of §5.1, p. 98

8) Let f be continuous on $[a,b]$ and suppose that for $x \in (a,b)$ we have

$$\limsup_{\epsilon \to 0^+} \frac{f(x+\epsilon) - f(x)}{\epsilon} \geq 0.$$

then for all $x, y \in [a,b]$ with $x < y$ we have $f(x) \leq f(y)$. (*Hint:* suppose $\delta > 0$ and let $g(x) = f(x) + \delta x$. Prove the claim for g then make an inference about f.)

9) Let f be a non-decreasing function on $[a,b]$. Then f is differentiable almost everywhere with derivative f'. [5] Moreover

$$\int_{[a,b]} f' \, dm \leq f(b) - f(a).$$

[5] H. L. Royden, *Real Analysis* 3rd ed., Theorem 3 of §5.1, p. 100. The proof uses Vitali's covering lemma.

10) Let V be a vector space and $E \subset V$ be a convex subset. We say that $x \in E$ is an *extreme point* of E if whenever we write

$$x = \lambda y + (1 - \lambda) z$$

with $y \neq z \in E$ we must have either $\lambda = 0$ or $\lambda = 1$.

a) Let $1 < p < \infty$ and $B \subset L^p[0,1]$ be the closed unit ball $B = \{x \mid \|x\| \leq 1\}$. Let $S = \{x \mid \|x\| = 1\}$ be the unit sphere. Show that every point of S is an extreme point of B and only these points are extreme points.

b) Let $B \subset L^\infty[0,1]$ be the closed unit ball $B = \{x \mid \|x\| \leq 1\}$. Show that x is an extreme point of B if and only if $|x(t)| = 1$ for almost all $t \in [0,1]$.

c) Let $B = \{x \in L^1[0,1] \mid \|x\| \leq 1\}$. Show that B has no extreme points.

11) Let $c_0 = \{(x_n)_n \mid \lim_n |x_n| = 0\}$ and for $x \in c_0$ let $\|x\|_\infty = \sup_n |x_n|$. Let $\ell^1 = \{(x_n) \mid \sum_n |x_n| < \infty\}$ and for $x \in \ell^1$ let $\|x\|_1 = \sum_n |x_n|$. Let $\ell^\infty = \{(x_n)_n \mid \sup_n |x_n| < \infty\}$ and for $x \in \ell^\infty$ let $\|x\|_\infty = \sup_n |x_n|$.

Show that

a) if $y \in \ell^1$, $x \in c_0$, and we let $\Lambda_y(x) = \sum_n x_n y_n$, then $\Lambda_y \in c_0^*$, $\|\Lambda_y\| = \|y\|_1$, and every $\Lambda \in c_o^*$ is Λ_y for a unique $y \in \ell^1$.

b) if $y \in \ell^\infty$ and $x \in \ell^1$ and we let $\Lambda_y(x) = \sum_n x_n y_n$, then $\Lambda_y \in \ell^{1*}$, $\|\Lambda_y\| = \|y\|_\infty$ and for every $\Lambda \in \ell^{1*}$ there is a unique $y \in \ell^\infty$ such that $\Lambda = \Lambda_y$.

12) Let $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ and $e_n : \mathbb{T} \to \mathbb{C}$ be defined by $e_n(z) = z^n$ for $n \in \mathbb{Z}$. The functions in the linear span of $\{e_n\}_n$ are called trigonometric polynomials. Let $C(\mathbb{T})$ be the Banach space of continuous complex valued functions on $\mathbb{T}$ with the norm $\|f\| = \sup_{z \in \mathbb{T}} |f(z)|$. Show that the trigonometric polynomials are dense in $C(\mathbb{T})$. (Theorem 4.15 in Rudin)

13) Let X and Y be Banach spaces and $T : X \to Y$ a linear map. The graph of T, $\Gamma(T)$, is the subspace $\{(x, Tx) \mid x \in X\} \subset X \oplus Y$. We make $X \oplus Y$ a normed space with the norm $\|x \oplus y\| = \|x\| + \|y\|$. We sat that the graph of t is closed if $\Gamma(T)$ is a closed subspace of $X \oplus Y$. Prove the closed graph theorem which assert that T is continuous if and only if its graph is closed.(6)

[6] See Rudin Exercise 5.16.

14) Let $A = (a_{ij})_{ij=1}^{\infty}$ be a matrix with complex entries. If $s = (s_1, s_2, s_3, \dots)$ is a sequence of complex numbers we let $\sigma = As$ be the sequence $(\sigma_1, \sigma_2, \sigma_3, \dots)$ whose i^{th} entry is

$$\sigma_i = \sum_{j=1}^{\infty} a_{ij} s_j.$$

Show that A transforms convergent sequences s to convergent sequences σ with the same limit if and only if the following three conditions are satisfied.

a) for all j, $\lim_{i \to \infty} a_{ij} = 0$

b) $\sup_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}| < \infty$

c) $\lim_{i \to \infty} \sum_{j=1}^{\infty} a_{ij} = 1.$

Show that $a_{ij} = \begin{cases} \frac{1}{i+1} & \text{if } 1 \le j \le i \\ o & \text{otherwise} \end{cases}$ satisfies the conditions, as does the matrix $a_{ij} = (1 - r_i) r_i^j$ where $0 < r_i < 1$ and $\lim_{i \to \infty} r_i = 1$. For each of these A's, give an example of a sequence s which doesn't converge but $\sigma = As$ does.(7)

[7] See Rudin Exercise 5.15.

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