$f: X \to Y$ is measurable and $g: Y \to Z$ is continuous, then $g \circ f$ is measurable.

Proof: Let $O \subseteq Z$ be open, then $g^{-1}(O) \subseteq Y$ is open and so $f^{-1}(g^{-1}(O)) \in \mathfrak{M}$. But $(g \circ f)^{-1}(O) = f^{-1}(g^{-1}(O))$. $\square$

Theorem

Let $(X, \mathfrak{M})$ be a measurable space and suppose $u, v: X \to \mathbb{R}$ are measurable functions. Suppose $\Phi: \mathbb{R}^2 \to Y$ is continuous where Y is a topological space. Then $h: X \to Y$ given by $h(x) = \Phi(u(x), v(x))$ is measurable.

Proof: Let $f(x) = (u(x), v(x))$. It suffices to show that f is measurable. Let $R \subseteq \mathbb{R}^2$ be the open rectangle $R = (a,b) \times (c,d)$. Then $f^{-1}(R) = u^{-1}(a,b) \cap v^{-1}(c,d) \in \mathfrak{M}$. Let $V \subseteq \mathbb{R}^2$ be an arbitrary open set.

(10)

Let $V_0 = \{(x,y) \in V \mid x, y \in \mathbb{Q}\}$ then V_0 is a countable dense sub set. (This reduces to showing that every open interval in $\mathbb{R}$ contains a rational number. This property goes back to how the real numbers are constructed.) Given $x \in V_0$ let $d(x, V^c)$ be the distance from x to the complement of V, and $R_x = (x_1 - \frac{d}{\sqrt{2}}, x_1 + \frac{d}{\sqrt{2}}) \times (x_2 - \frac{d}{\sqrt{2}}, x_2 + \frac{d}{\sqrt{2}})$ where $d = d(x, V^c)$ and $x = (x_1, x_2)$. Let $v \in V$ and choose $x \in V_0$ so that $d(x, v) < \lambda d(v, V^c)$ (with $\lambda < 0.4$). Then $d(x, V^c) > (1-\lambda)\, d(v, V^c)$.

So $d(x,v) < \frac{1-\lambda}{\lambda\sqrt{2}} d(x,v) < \frac{1-\lambda}{\sqrt{2}} d(v, V^c) < \frac{d(x, V^c)}{\sqrt{2}}$

Thus $v \in R_x$. So $V = \bigcup_{x \in V_0} R_x$ is a countable union of open rectangles. Then $f^{-1}(V) = \bigcup_{x \in V_0} f^{-1}(R_x) \in \mathfrak{M}$, and thus f is measurable.

Proposition

Suppose $(X, \mathcal{M})$ is a measurable space and $f: X \to \mathbb{R}$. f is measurable if and only if $\forall a \in \mathbb{R}$ $f^{-1}(a, \infty) \in \mathcal{M}$. Similarly, if $f: X \to \overline{\mathbb{R}}$ we require $f^{-1}(a, \infty] \in \mathcal{M}$.

Proof: Suppose $\forall a \in \mathbb{R}$ $f^{-1}(a,\infty) \in \mathcal{M}$. For $b \in \mathbb{R}$, $[b, \infty) = \bigcap_n (b - n^{-1}, \infty)$ so

$f^{-1}([b,\infty)) = \bigcap_n f^{-1}(b - n^{-1}, \infty) \in \mathcal{M}$. Thus

$f^{-1}((-\infty, b)) = [f^{-1}([b,\infty))]^c \in \mathcal{M}$. Thus

$f^{-1}((a,b)) = f^{-1}(-\infty\, b) \cap f^{-1}(a,\infty) \in \mathcal{M}$.

Fact: every open set in $\mathbb{R}$ is a countable union of open intervals. (proof similar to one above)

Fact: every open set in $\mathbb{R}$ is a countable disjoint union of open intervals.

In either case we have if $O = \bigcup_n I_n$ with each I_n an open interval $f^{-1}(O) = \bigcup_n f^{-1}(I_n) \in \mathcal{M}$. The other direction follows from the definition.

Corollary

Suppose $(X, \mathfrak{M})$ is a measurable space, and $u, v : X \to \mathbb{R}$ are measurable.

(a) $u+v,\ uv : X \to \mathbb{R}$ are measurable

$u + iv : X \to \mathbb{C}$ is measurable

(b) If $f, g : X \to \mathbb{C}$ are measurable then so are $f+g,\ fg,\ Re(f), Im(f)$

(c) If $E \subseteq X$ then $E \in \mathfrak{M}$ if and only if χ_E is measurable

(d) If $f : X \to \mathbb{C}$ is measurable then we can write $f = \alpha |f|$ with $\alpha, |f|$ measurable and $|\alpha| = 1$.

Remark (before proof of (d))

Suppose $(X, \mathfrak{M})$ is a measurable space and $E \in \mathfrak{M}$. Let $\mathfrak{M}_E = \{F \cap E \mid F \in \mathfrak{M}\}$. Then $\mathfrak{M}_E$ is a σ-algebra of subsets of E. $E \setminus (F \cap E) = E \cap F^c$. Suppose we can write $X = E_1 \cup \cdots \cup E_n$ of measurable sets with the E_i's pairwise disjoint. $\mathcal{M}_{E_i} \subseteq \mathcal{M}$.

If $F \subseteq X$ and $F \cap E_i \in \mathcal{M}_i$ then

$$F = (F \cap E_1) \cup \cdots \cup (F \cap E_n) \in \mathcal{M}.$$

If $f_i : E_i \to Y$ (a topological space) and f_i is $\mathcal{M}_{E_i}$- measurable then we can define a measurable function $f : X \to Y$ by

$$f(x) = f_i(x) \quad \text{if } x \in E_i.$$

$$f^{-1}(O) = \bigcup_{i=1}^{n} f_i^{-1}(O) \in \mathcal{M}.$$

Proof of (d)

Let $E = f^{-1}(\{0\}) \in \mathcal{M}$. For $x \in E^c$ let $\alpha(x) = \dfrac{f(x)}{|f(x)|}$ and for $x \in E$ let $\alpha(x) = 0$. $\alpha|_E$ is $\mathcal{M}_E$-measurable. $\alpha|_{E^c}$ is $\mathcal{M}_{E^c}$ measurable because it is a product of measurable functions. Thus α is measurable. $|f|$ is also measurable and $\alpha |f| = f$.

Theorem Let X be a set and $\mathcal{F}$ be a collection of sets. Then there is a smallest

σ-algebra containing $\mathcal{F}$.

Proof: Let $P(X)$ be the collection of all subsets of X, $P(X)$ is called the power set of X. $P(X)$ is a σ-algebra and $\mathcal{F} \subseteq P(X)$. Let Ω be the class of all σ-algebras containing $\mathcal{F}$. $P(X) \in \Omega$, so Ω is not empty.

Let $\mathcal{M} = \bigcap_{\mathcal{N} \in \Omega} \mathcal{N}$. $\mathcal{F} \subseteq \mathcal{M}$. So let us show that $\mathcal{M}$ is a σ-algebra. If $\mathcal{N} \in \Omega$ then $X \in \mathcal{N}$ so $X \in \bigcap_{\mathcal{N} \in \Omega} \mathcal{N}$. Hence $X \in \mathcal{M}$. If $A \in \mathcal{M}$ then $A \in \mathcal{N}$ for all $\mathcal{N} \in \Omega$ and $A^c \in \mathcal{N}$ $\forall \mathcal{N} \in \Omega$ so $A^c \in \mathcal{M}$. If $\{A_n\}_n \subseteq \mathcal{M}$ then $\{A_n\}_n \subseteq \mathcal{N}$ $\forall \mathcal{N} \in \Omega$ so $\bigcup_n A \in \mathcal{N}$ $\forall \mathcal{N} \in \Omega$. Thus $\bigcup_n A_n \in \mathcal{M}$. Hence $\mathcal{M}$ is a σ-algebra. Since $\mathcal{M}$ is contained in every σ-algebra containg $\mathcal{F}$, $\mathcal{M}$ must be the smallest σ-algebra containing $\mathcal{F}$.

Borel Sets

Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B}$ be the smallest σ-algebra containing $\mathcal{T}$. Elements of $\mathcal{B}$ are the Borel subsets of X. $\mathcal{B}$ is the σ-algebra of Borel sets.

- if $\{A_n\}_n \subseteq \mathcal{T}$ then $\bigcap_n A_n \in \mathcal{T}$.
  a countable intersection of open sets is called a G_δ set.
- if $\{C_n\}_n$ is a countable collection of closed sets $\bigcup_n C_n \in \mathcal{B}$ and is called a F_σ set.
- every open set and every closed set is a Borel set
- $[a,b)$, $(a,b]$ are Borel sets of $\mathbb{R}$.
- $\mathbb{Q} \in F_\sigma$, $\mathbb{R} \setminus \mathbb{Q} \in G_\delta$
- in a metric space every closed set is a G_δ set and every open set is a F_σ set.

Borel Functions

Suppose X & Y are topological spaces and

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$f: X \to Y$. If for every open set $O \subseteq Y$ $f^{-1}(O)$ is a Borel subset of X, we say that f is a Borel function.

Every continuous function is a Borel function (or a Borel measurable function)

Theorem

Let $(X, \mathcal{M})$ be a measurable space and Y a topological space and $f: X \to Y$.

(a) Let $\mathcal{N} = \{E \subseteq Y \mid f^{-1}(E) \in \mathcal{M}\}$. Then $\mathcal{N}$ is a σ-algebra.

(b) If $E \subseteq Y$ is a Borel set and f is $\mathcal{M}$-measurable then $f^{-1}(E) \in \mathcal{M}$

(c) If Z is a topological space and $g: Y \to Z$ is a Borel function and $f: X \to Y$ is measurable then $g \circ f$ is measurable.

Proof: (a) $f^{-1}(Y) = X$, $f^{-1}(A^c) = [f^{-1}(A)]^c$, $f^{-1}(\bigcup_n A_n) = \bigcup_n f^{-1}(A_n)$

(b) If f is measurable then $\{E \subseteq Y \mid f^{-1}(E) \in \mathfrak{M}\}$ is a σ-algebra on Y containing the open sets and thus the Borel sets

(c) if $A \subseteq Z$ is open then $g^{-1}(A)$ is Borel and $f^{-1}(g^{-1}(A)) \in \mathfrak{M}$. Hence $(g \circ f)^{-1}(A) \in \mathfrak{M}$ and so $g \circ f$ is measurable

Limsup & Liminf

Let $\{a_n\}_n \subseteq \overline{\mathbb{R}}$ and $b_n = \sup_{k \geq n} a_k \in \overline{\mathbb{R}}$.

$b_1 \geq b_2 \geq b_3 \geq \cdots$ is a decreasing sequence

let $\limsup_n a_n = \inf_n b_n = \inf_n \left(\sup_{k \geq n} a_k \right)$.

Let $T = \{t \in \overline{\mathbb{R}} \mid \exists \{a_{n_k}\}_{k=1}^{\infty}$ a subsequence converging to $t\}$

$\limsup_n a_n = \sup_{t \in T} t$. Let $\varepsilon > 0$

$\{n \mid a+\varepsilon < a_n\}$ finite $\Leftrightarrow$ for n large enough $b_n \leq a+\varepsilon$

$\Leftrightarrow \inf_n b_n \leq a+\varepsilon$

$\{n \mid a-\varepsilon < a_n\}$ infinite $\Leftrightarrow \forall n \; b_n > a-\varepsilon$

$\Leftrightarrow \inf_n b_n \geq a-\varepsilon$

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Thus $\forall \varepsilon>0$ $\{n \mid a+\varepsilon < a_n\}$ is finite and

$\{n \mid a-\varepsilon < a_n\}$ is infinite $\Longleftrightarrow$

$\forall \varepsilon>0 \quad a-\varepsilon \leq \limsup_n a_n \leq a+\varepsilon$

$\Longleftrightarrow \quad \limsup_n a_n = a$

Let $c_n = \inf_{k\geq n} a_k \qquad c_1 \leq c_2 \leq c_3 \leq \cdots$

let $\liminf_n a_n = \sup_n c_n$.

Facts

(a) $\liminf_n a_n \leq \limsup_n a_n$

(b) $\limsup_n(-a_n) = -\liminf_n a_n$

(c) $\limsup_n a_n = \liminf_n a_n \Leftrightarrow \lim_n a_n$ exists and equals these numbers

(d) $\limsup_n a_n+b_n \leq \limsup_n a_n + \limsup_n b_n$

Provided there are no sums of the form $\infty - \infty$

(e) if $a_n \leq b_n$ then $\liminf_n a_n \leq \liminf_n b_n$ and $\limsup_n a_n \leq \limsup_n b_n$