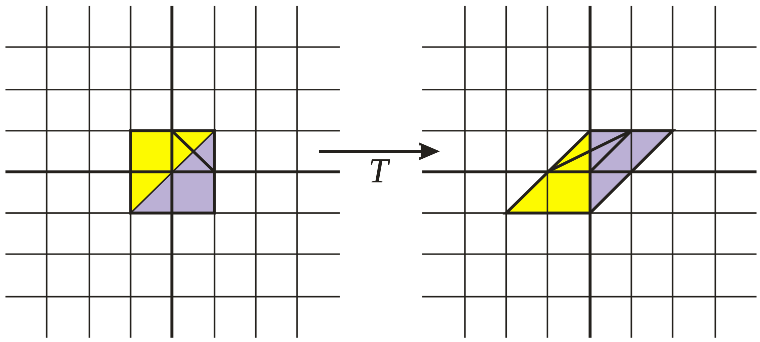
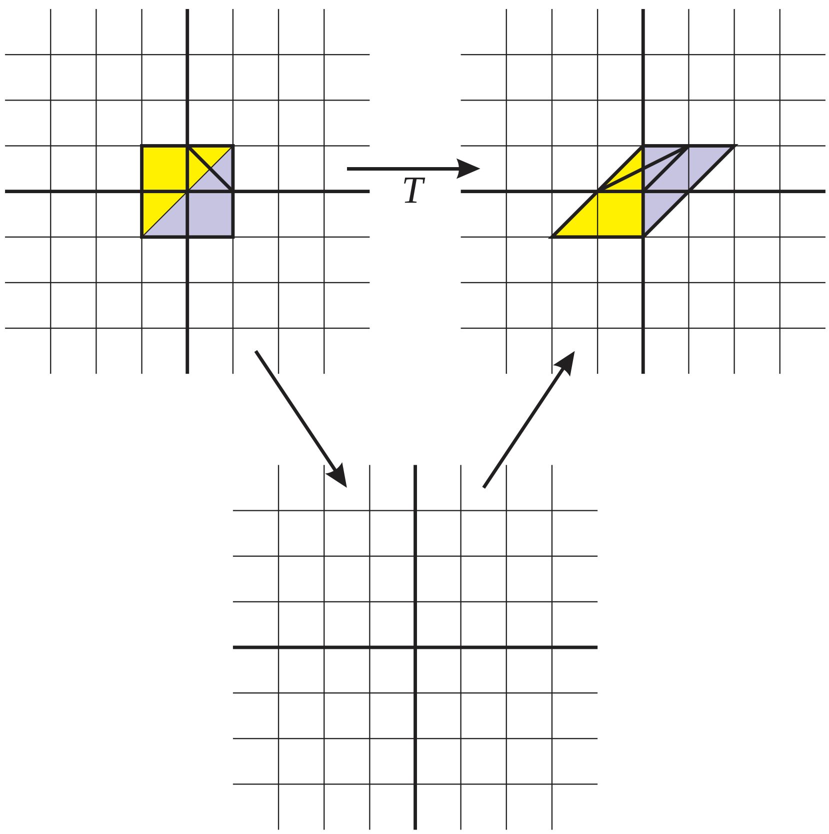
Example 1--geometry

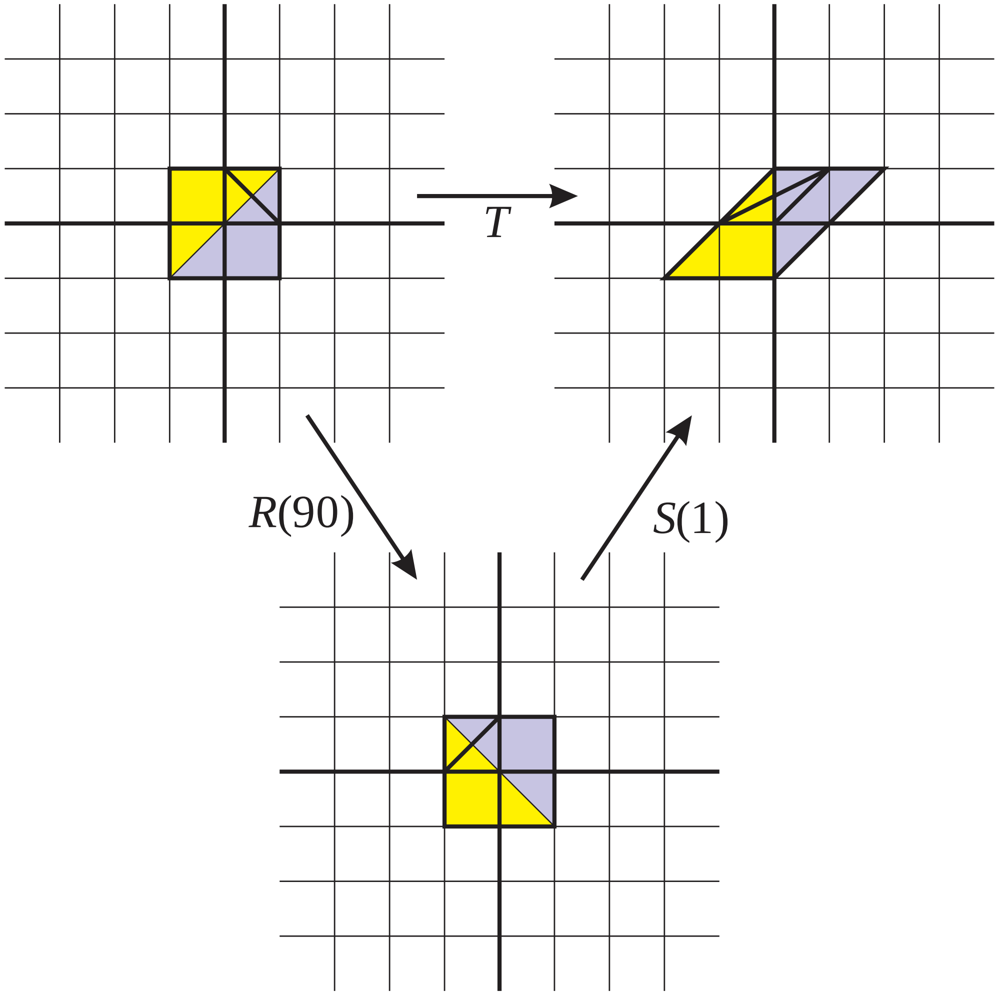
Here's a simple linear transformation. It can be constructed out of basic transformations in many ways.



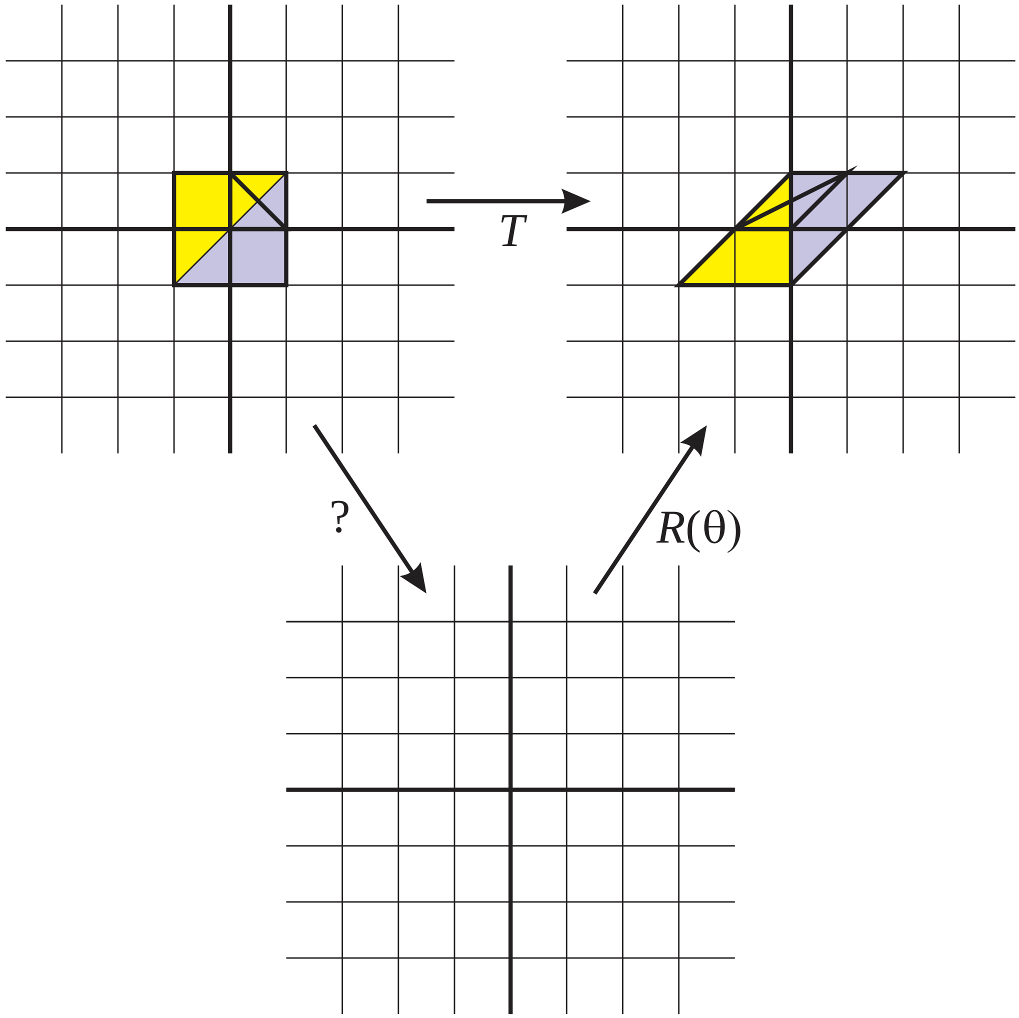
One of these uses only two steps. Can you find it?



That’s not so hard—rotate first and then shear

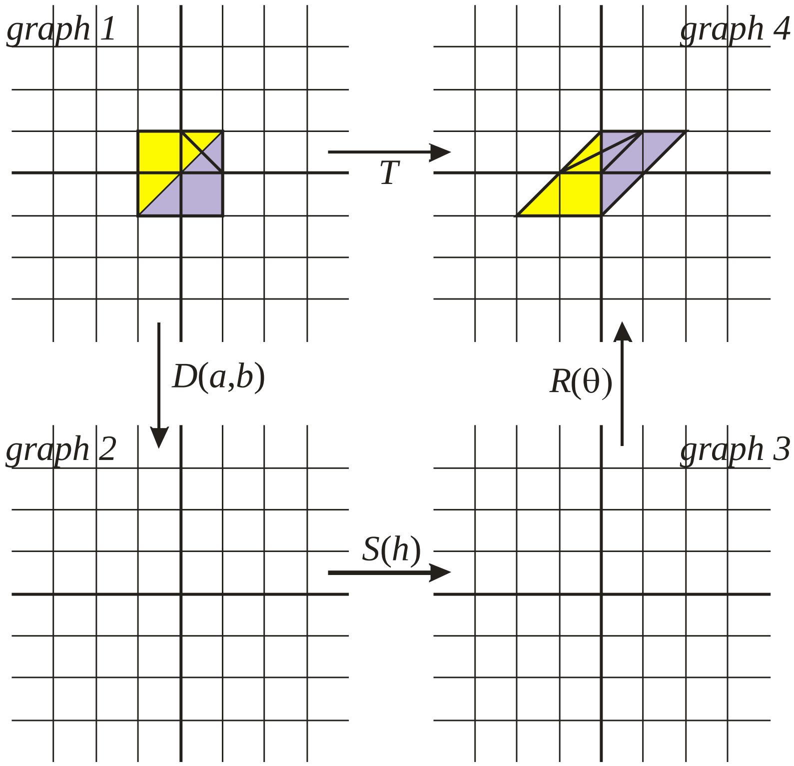


Now suppose I want you to do the rotation last. Can you do that in two steps?



The answer is no. There are some nice simple arguments that this cannot be done (using the basic transformations). Can you find one?

But it *can* be done in three steps. Below I give you a possible recipe—a dilation followed by a shear followed by a rotation? Can you find the parameters *a*, *b*, *h* and *θ* for the three steps? For ease of discussion I have numbered the four graphs in order of the transformation steps.

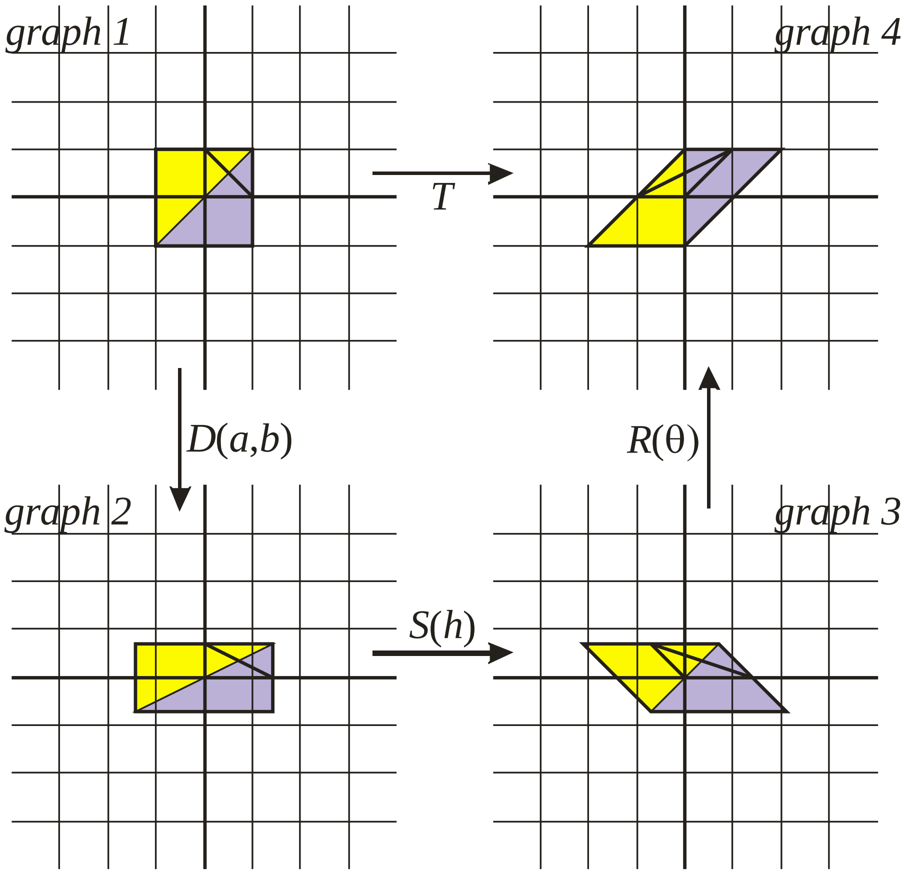


Where should we begin? In fact there are two reasonable places to start, at the beginning and at the end—with the dilation and with the rotation. It turns out that both of these require some preliminary investigation of the "invariance" properties of the basic transformations. Dilations, shears and rotations all change the object in different ways but each type of transformation has some geometric properties that are left unchanged, and it is these "invariants" that will hold the key to many of our arguments. You'll see what I mean as we go along.

It turns out to be quite a bit easier to make arguments if we have candidate diagrams in the two intermediate grids. The students will have access to an app that allows them to experiment quickly with different parameters, but few of my grade 10 students had much experience with geometric arguments so it was helpful for them to have diagrams of the intermediate graphs at the beginning. And that's what I now always give them. After all, most of them are in the early stages of practicing "mathematical thinking."

So here's how I present the problem to the students. The form of the different graphs gives them some valuable insights into the "invariance" properties of the transformations that I was talking about.

Can you find the parameters *a*. *b*, *h* and *θ*?

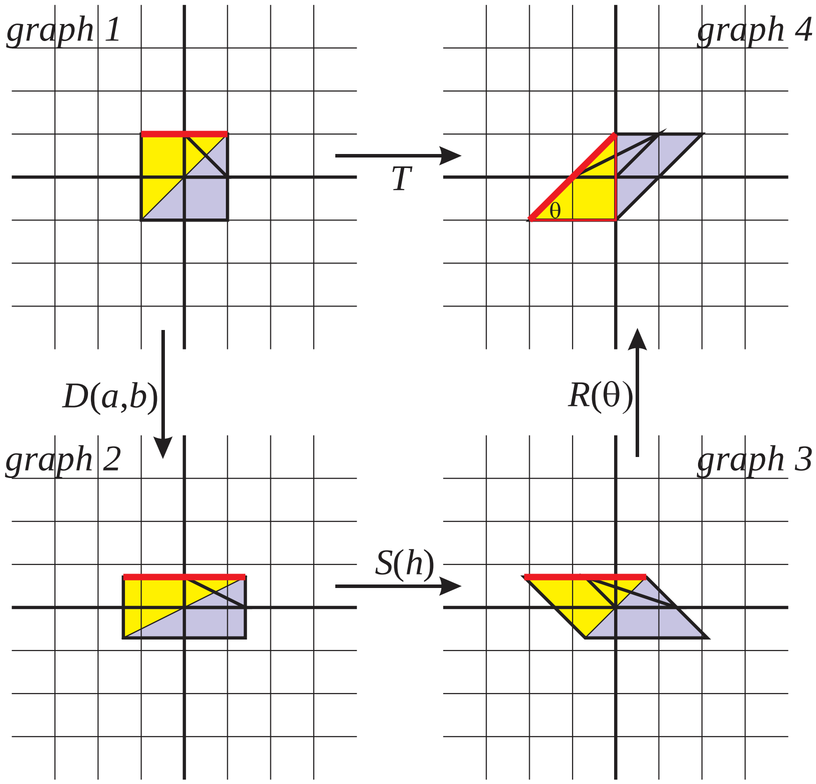


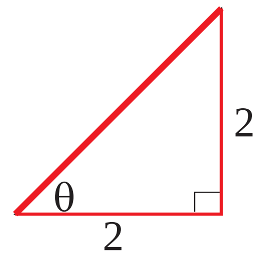
This problem lends itself well to small-group work. In fact this in one of the easiest examples and my objective is that every student should be able to master the algorithm at this level of difficulty. Other examples are more challenging than this.

I have just called the method of solution an "algorithm" and it is and it isn't. It *is* algorithmic in the sense that there is a way of decomposing the argument that will in principle work for all examples (for the decomposition dilation followed by shear followed by rotation). But it isn't really algorithmic in that technical difficulties can sometimes arise requiring a new insight.

Finding θ.

Given the full diagram, the answer seems clear—the rotation is $\theta = 45°$ counterclockwise. But a careful argument is needed. Fasten attention on the top of the box. I have marked in red its footprints in the four graphs.





The argument we need to make is that in going from graph 3 to graph 4 the red line rotates through 45º. But can we be sure of that? In graph 4 it is indeed angled at 45º—we can be certain of this because the target image in graph 4 is anchored at grid points (and this will always be the case for us) and that gives us a 45-45-90 triangle. But what about the graph 3 image? For our argument to hold we need to know that the red line in graph 3 is horizontal. Well, that certainly appears to be the case. But can we be sure?

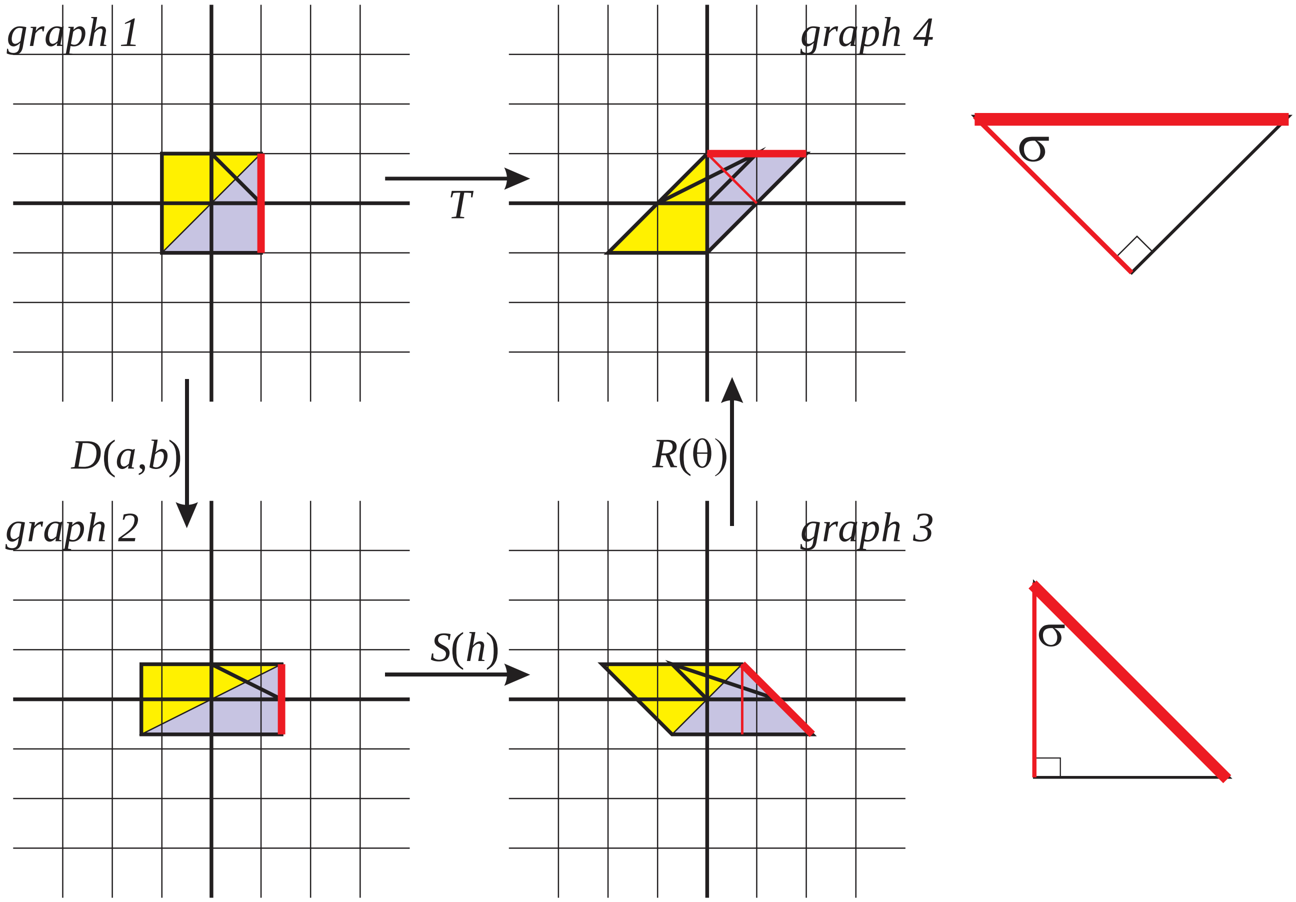
The answer is, yes we can—and it's important to be able to make that argument. The key is found in the invariance properties of the component transformations.

Start at graph 1. The top of the box is horizontal. Again we know that because in graph 1 the box is anchored at grid points. Moving on, in graph 2 the top of the box is still horizontal. We don't have grid points anymore so how do we know that? We know that because dilations leave horizontal lines horizontal (though they might change height and length). Moving on, in graph 3 the top of the box is *still* horizontal. Again we don't have grid points so the reason we know that is because *the shear leaves horizontal lines horizontal* (though it moves them sideways).

These are significant and valuable invariance properties—dilations and (horizontal) shears both *preserve* "*horizontality*"! That's the property that gives us the proof that the rotation angle θ is 45º.

Finding h.

This problem can be solved working directly with h or with the shear angle σ. My view is that it's a bit easier to work with σ. We need to see what happens to a vertical line in graph 2 and I choose to work with the right-hand side of the starting box. I mark (in red) its footprints in the other graphs.



Okay—now for this already familiar chain of argument. The heavy red line is vertical in graph 1 (grid points), therefore it's vertical in graph 2 (dilations preserve verticality). In the passage to graph 3 it gets rotated by the shear angle σ, displayed in the graph-3 blow-up as one of the angles in a right-angled triangle. But what angle is it? That's hard to say because we are not anchored to grid points in graph 3. But if we look at that triangle in graph 4 we see that it is isosceles (the two legs are both diagonals of a 1×1 square). That tells us that $\sigma = 45$. Thus the shear parameter is:

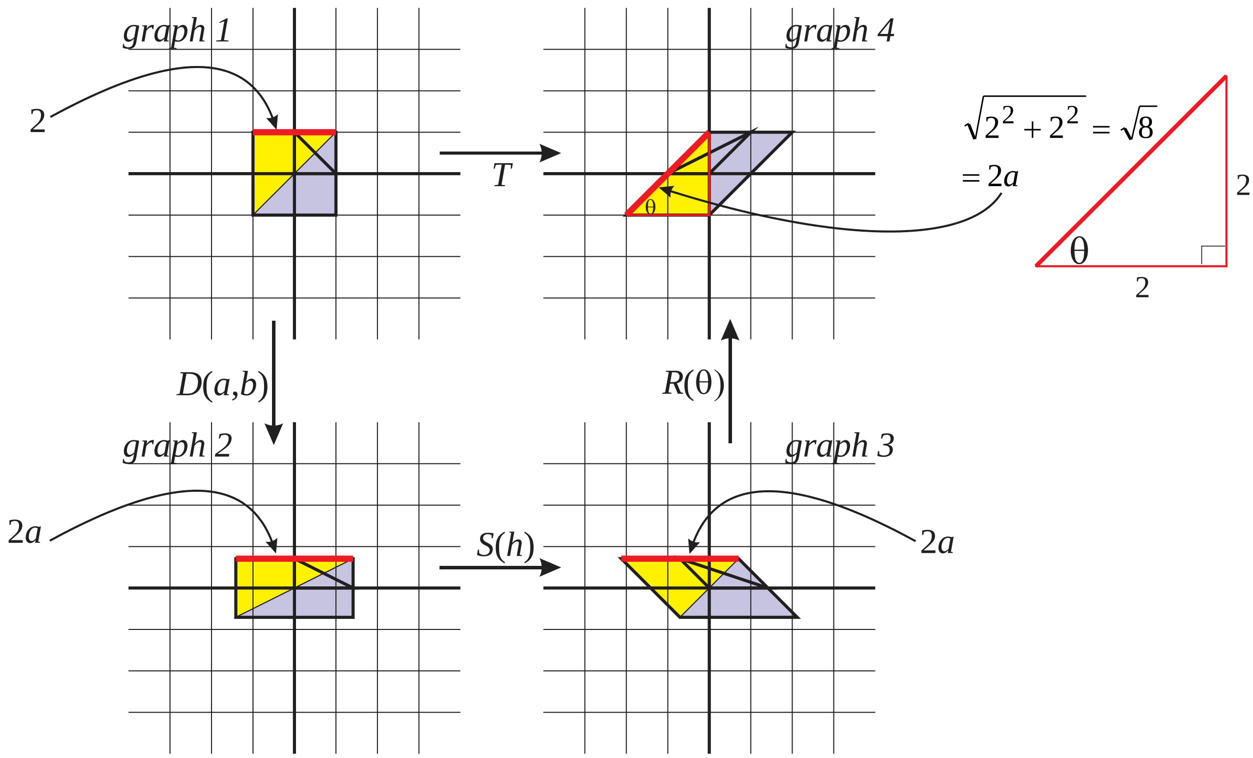
$$h = -\tan\sigma = -\tan 45 = -1$$

and we check that the sign of h really is negative.

Finally, we need to find the parameters a and b of the dilation D. These are found by comparing the width and height of the boxes in graphs 1 and 2. The problem is that there are no grid points to rely on in graph 2. To handle that, we will have to find a way of extracting the information from graph 4.

Finding a.

Follow the top of the box keeping track of its length. In graph 1 it has length 2; in graph 2 it has length $2a$.



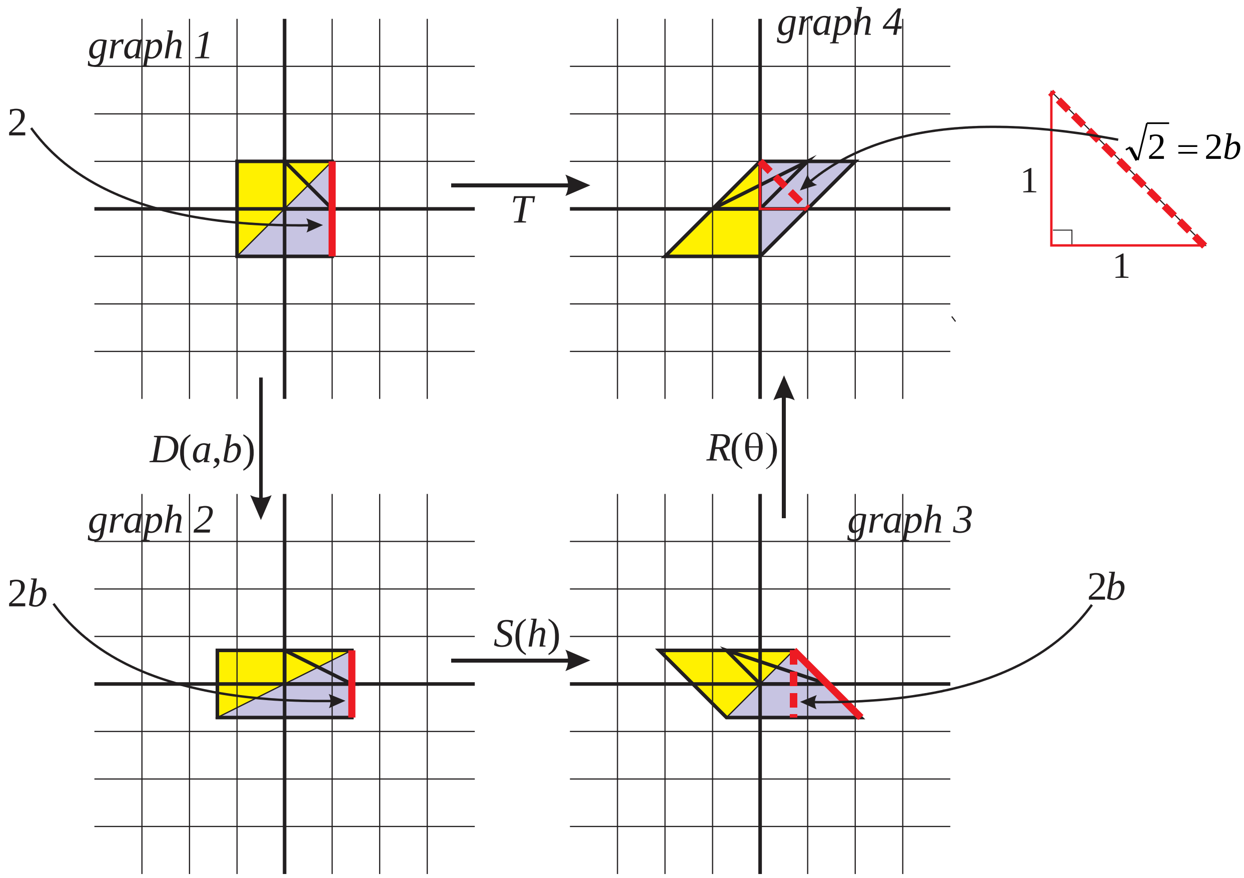
In graph 3 it still has length $2a$ because it was horizontal in graph 2 (already established) and the shear does not change the length of horizontal lines (it only moves them sideways). Finally, its footprint in graph 4 is also of length $2a$ because rotations do not change the length of any line.

But we can measure the length of the footprint in graph 4 (because we have the grid points we can rely on). It's the hypotenuse of a right-angled triangle with legs of length 2 so has length $\sqrt{8}$. This is $2a$. Thus

$$a = \frac{\sqrt{8}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

Finding b.

This time we need to keep track of the length of a vertical line as it jumps from graph 1 to graph 2. I choose the right-hand side of the box.

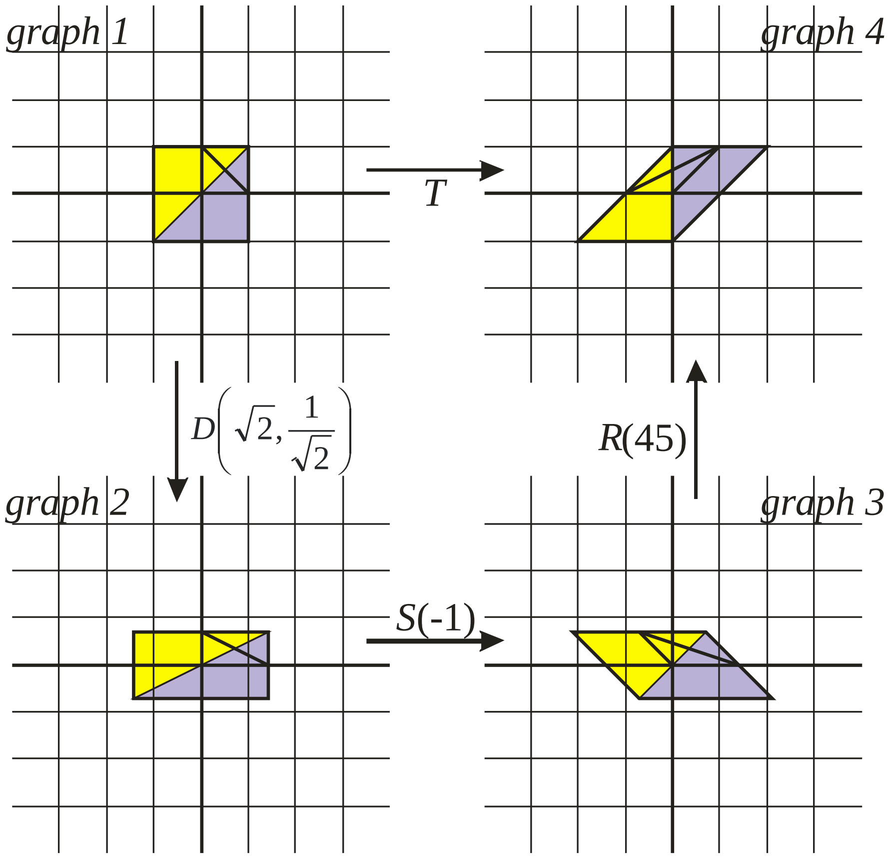


In graph 1 it has length 2; in graph 2 it has length 2*b*. Now in graph 3 we have to be careful—the right-hand side of the box (solid red) is no longer the same length—the shear changes the length of lines that are not horizontal. *However the vertical distance between horizontal lines does not change from graph 2 to graph 3* as the shear only moves horizontal lines sideways. Thus the dashed red line in graph 3 is of length 2*b*.

Finally the footprint of the dashed red line in graph 4 will have the same length 2*b*, and we can use the grid points to measure that. It's the hypotenuse of a right-angled triangle with legs of length 1 so has length $\sqrt{2}$. This is 2*b*. Thus

$$b = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

Putting everything together, we have the solution.



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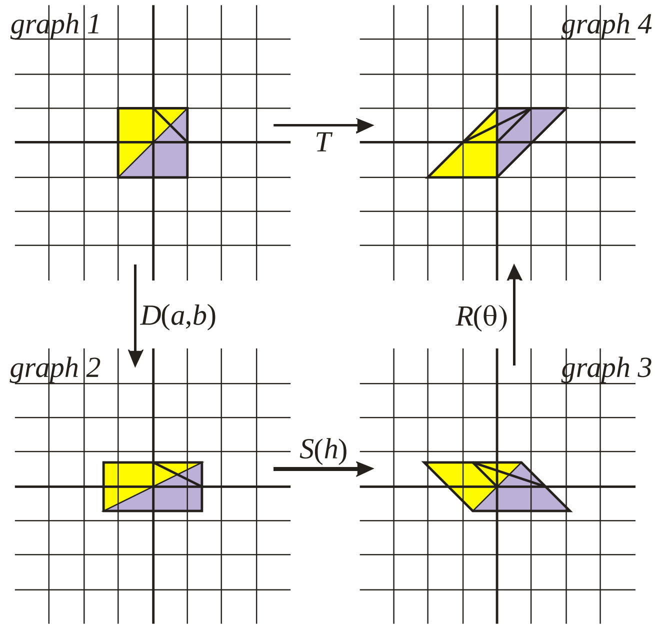
Example 1 algebra

Now let's solve Example 1 using our new algebraic tools.



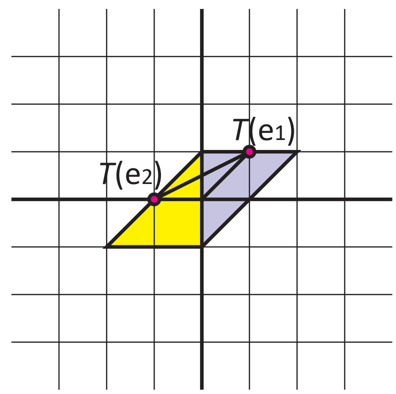
We need to find θ, a, b and h such that:

$$T = R(\theta) \circ S(h) \circ D(a,b)$$

$$[T] = [R(\theta)] \cdot [S(h)] \cdot [D(a,b)]$$

Note that when we move from the first *transformation* equation to the second *matrix* equation, the composition symbol $\circ$ is replaced by the multiplication sign. And I will remind you that the beginning transformation, in this case the dilation, is written "last," that is, on the right.

The basic images are $T(\mathbf{e}_1) = (1, 1)$ and $T(\mathbf{e}_2) = (-1, 0)$. The matrix equation is:



$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

To simplify notation I set $s = \sin\theta$ and $c = \cos\theta$:

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \cdot \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

Now multiply the matrices out. Let's start with the last two:

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \cdot \begin{bmatrix} a & hb \\ 0 & b \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} ca & chb - sb \\ sa & shb + cb \end{bmatrix}$$

Comparing each of the four entries, we get 4 equations in 4 unknowns:

(1) $1 = ca$

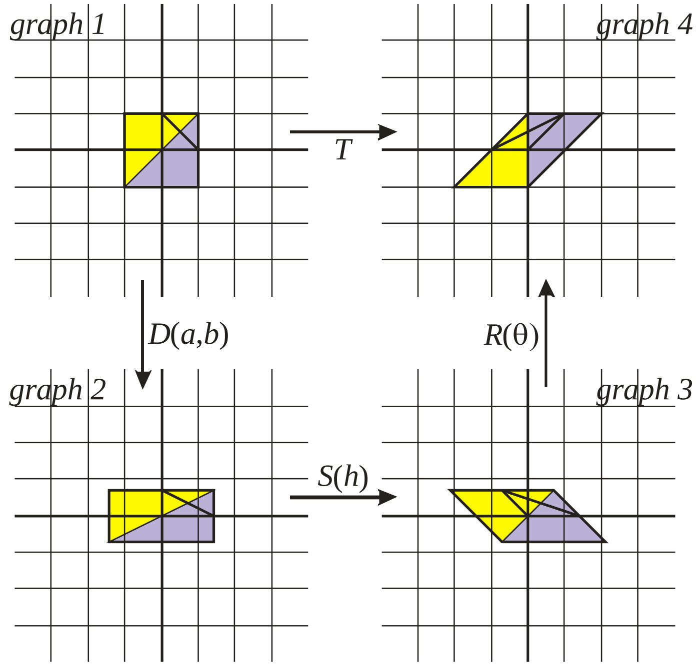
(2) $1 = sa$

(3) $-1 = chb - sb$

(4) $0 = shb + cb$

and we want to solve these for the four unknowns θ, a, b and h?

Solving the equations



(1) $1 = ca$

(2) $1 = sa$

(3) $-1 = chb - sb$

(4) $0 = shb + cb$

Divide (2) by (1) to get

$$\frac{sa}{ca} = \frac{1}{1} = 1$$

$$\frac{s}{c} = \frac{\sin\theta}{\cos\theta} = 1 \quad \text{(cancel the } a \text{ in top and bottom)}$$

$\tan\theta = 1$
$\theta = 45.$

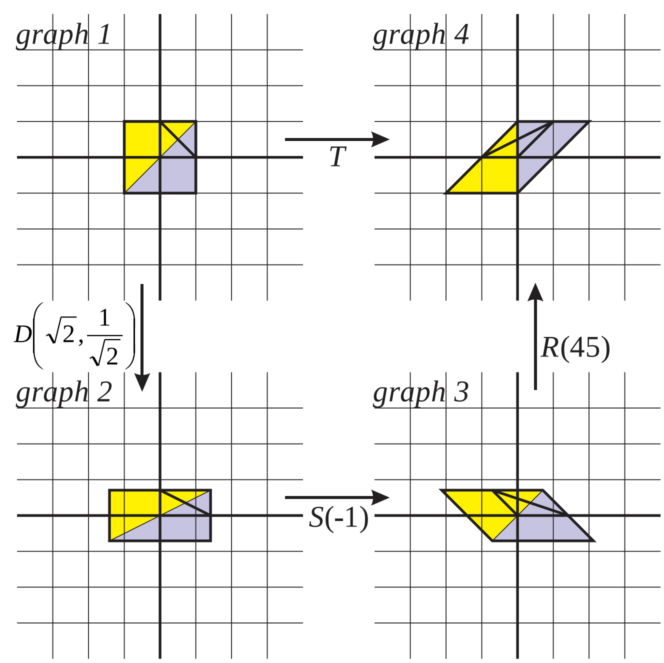
Check that this matches the diagram.

Hence $s = \sin\theta = \sin 45 = 1/\sqrt{2}$

$c = \cos\theta = \cos 45 = 1/\sqrt{2}$

From (2) we then get $a = \frac{1}{s} = \frac{1}{\sin\theta} = \sqrt{2}$,

Since $\sqrt{2} \approx 1.4$, this makes sense from the diagram.



From (4) we then get $shb = -cb$

$$h = \frac{-cb}{sb} = -\frac{c}{s} = -\frac{1/\sqrt{2}}{1/\sqrt{2}} = -1$$

And this also makes sense from the diagram. Finally, from (3):

$sb - chb = 1$

$$b = \frac{1}{s - ch} = \frac{1}{(1/\sqrt{2}) - (1/\sqrt{2})(-1)} = \frac{1}{2/\sqrt{2})} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

and this is close to 0.7 and that also makes sense from the diagram.

These are the same values we got with our geometric solution.