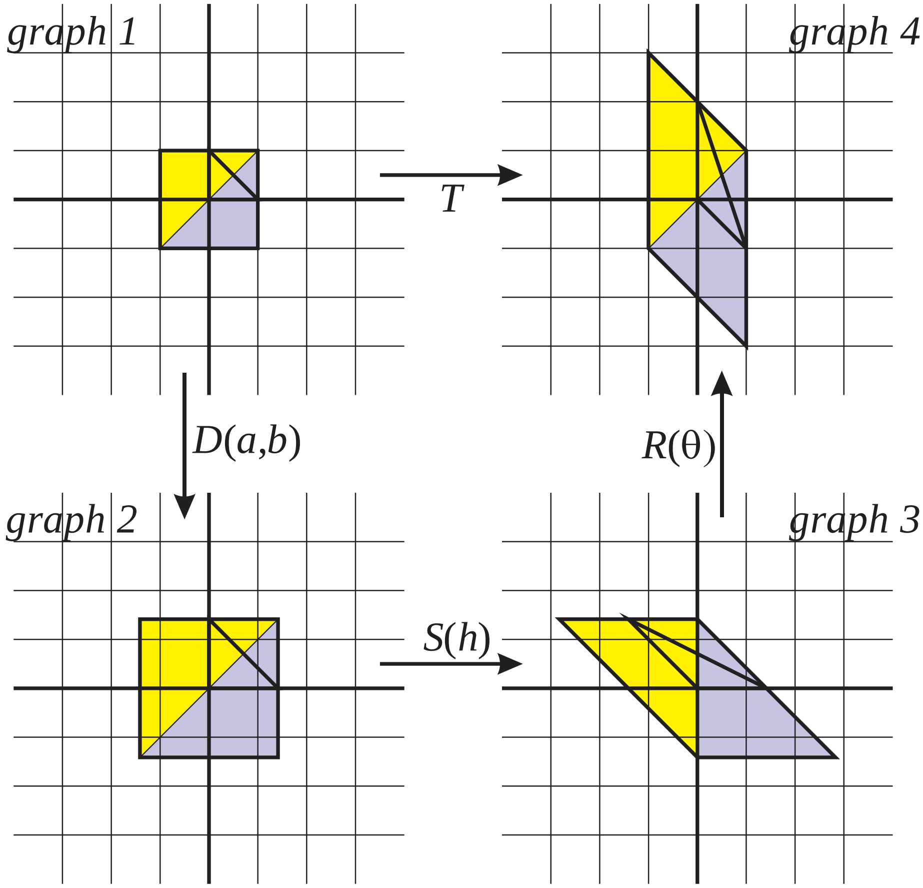
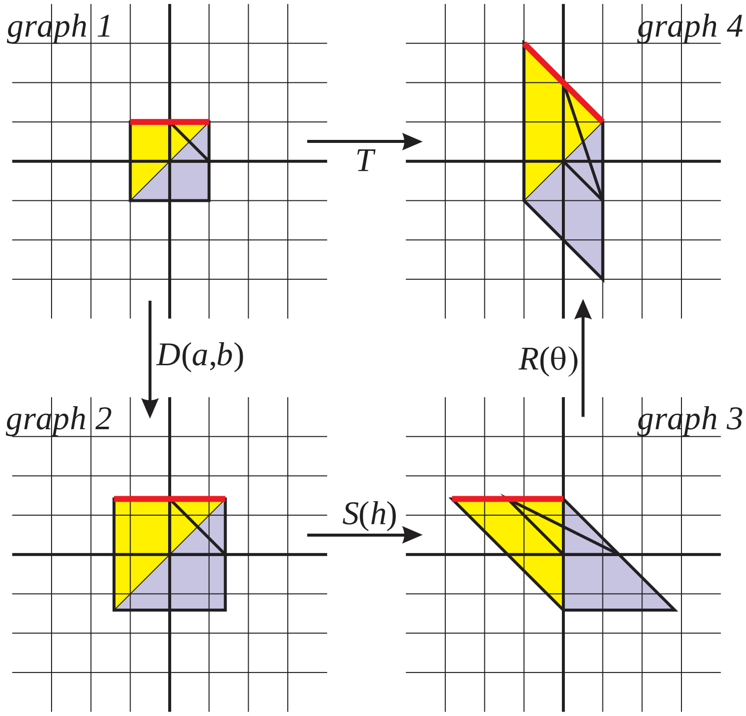
**Example 2 geometry**

The solution for this problem is very similar to what we have just seen in Example 1—thus it provides a good review and consolidation for the students. Find the parameters *a*, *b*, *h* and *θ*.



*Finding θ*
We use the same principle of horizontal invariance we used in Example1. Start at graph 1. The top of the box is horizontal (using the grid points). Since horizontal lines remain horizontal under dilation, the top of the box in graph 2 is still horizontal. Since the shear simply moves horizontal lines sideways, the top of the graph 3 box is still horizontal. Thus the angle $\theta$ of rotation is the angle its footprint in graph 4 makes with the horizontal and this is clearly 45º. Actually, the rotation from graphs 3 to 4 is clockwise so the angle is negative:

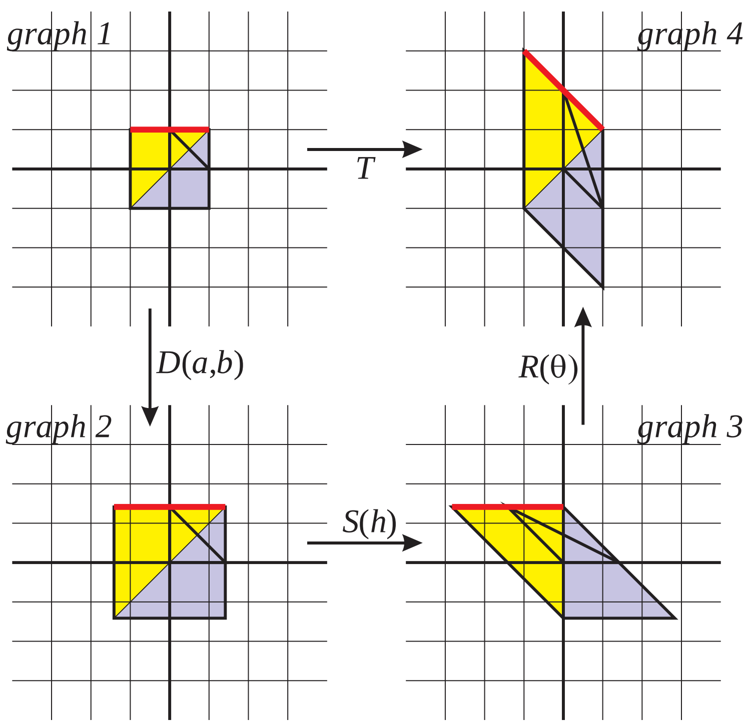
$$\theta = -45.$$




*Finding a*
We have to compare the lengths of a horizontal line in graphs 1 and 2. We use the same red line that served us in the $\theta$-analysis. Using the grid points, it has length 2 in graph 1 and thus its footprint in graph 2 has length 2$a$.

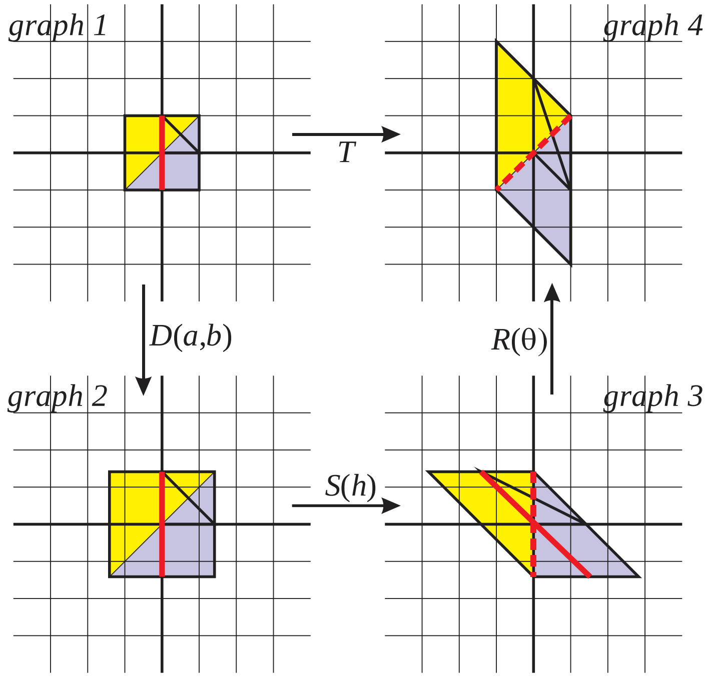
We now argue that its footprints in graphs 3 and 4 both have this same length 2$a$. First, since it is horizontal in graph 2 its length is unchanged by the shear and secondly, this length is again unchanged by the rotation. But in graph 4, the grid points give its length exactly as $2\sqrt{2}$ (twice the diagonal of a 1×1 square). Thus:

$$2a = 2\sqrt{2} \quad \Rightarrow \quad a = \sqrt{2}$$



*Finding b*

This time we need to keep track of the length of a vertical line as it jumps from graph 1 to graph 2. This time I choose the vertical centre-line of the square. Its length in graph 1 is 2, and thus its length in graph 2 will be 2*b*.



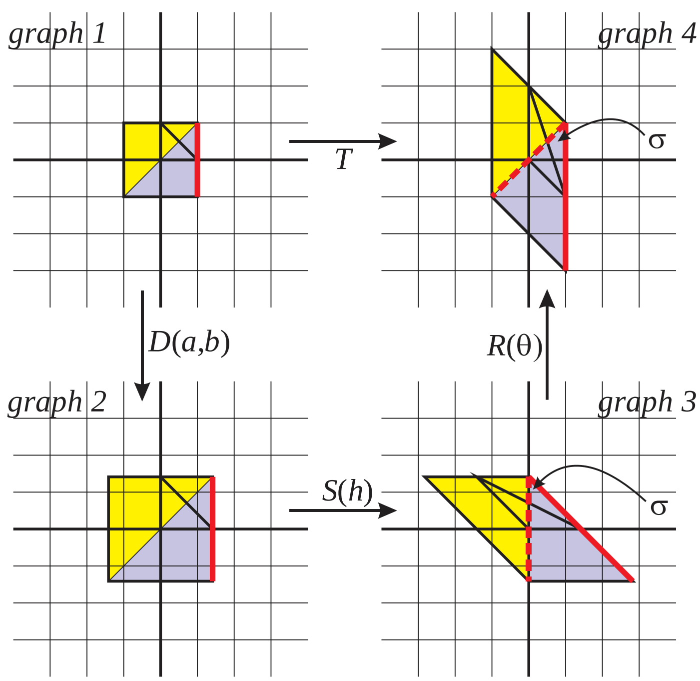

Now the footprint of this line in graph 3 is the solid red diagonal down the centre of the parallelogram, but it is longer than the graph-2 footprint. To find a line in graph 3 that has the same length as the red line in graph 2, we think of the graph-2 line as the perpendicular distance between the top and bottom of the square. Since these are horizontal lines the perpendicular distance between them will be the same in graph 3 and that's the length of the dashed red line. So it has length 2*b*. It's footprint in graph 4 has the same length (rotation) and clearly has length $2\sqrt{2}$. Thus:

$$2b = 2\sqrt{2} \quad \Rightarrow \quad b = \sqrt{2}$$

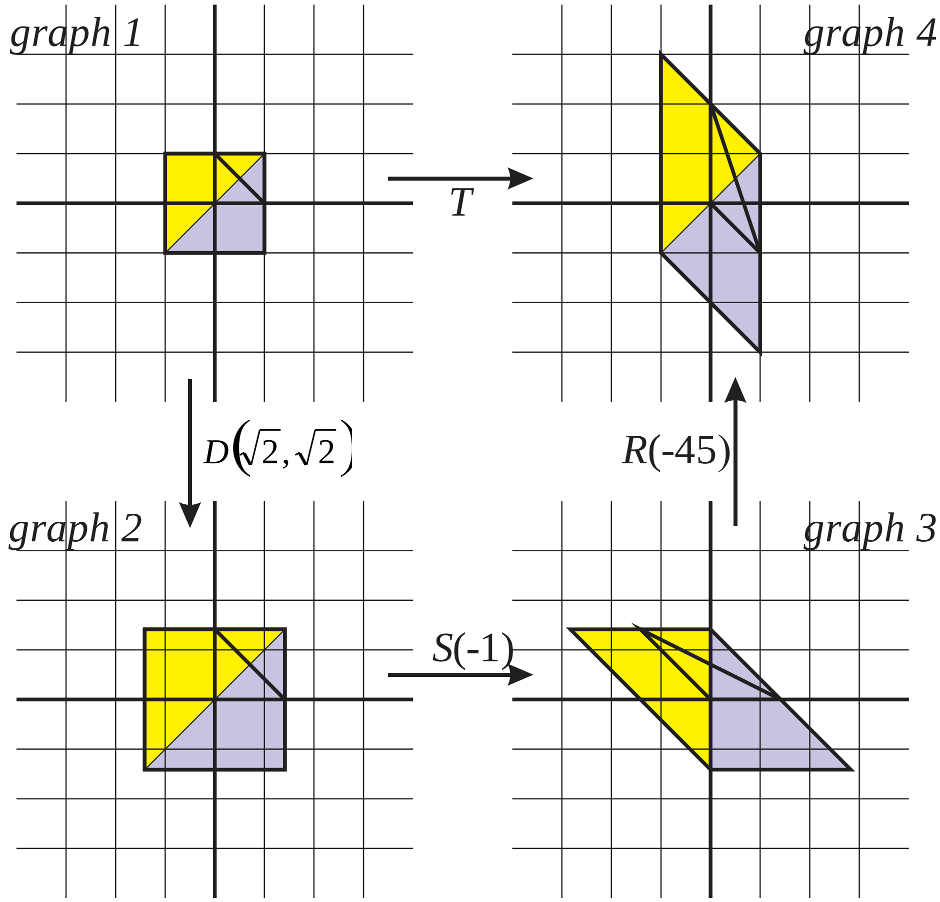
*Finding h*

We look for the shear angle $\sigma$. Consider the footprints of the right-hand edge of graph 1 in graphs 2, 3 and 4 (all solid red lines). The angle of rotation of this line from graph 2 to graph 3 is the shear angle $\sigma$. This angle can be seen in graph 4 to be $\sigma = 45^\circ$. Hence

$h = -\tan\sigma = -\tan 45 = -1$.



Putting everything together, we have the solution.



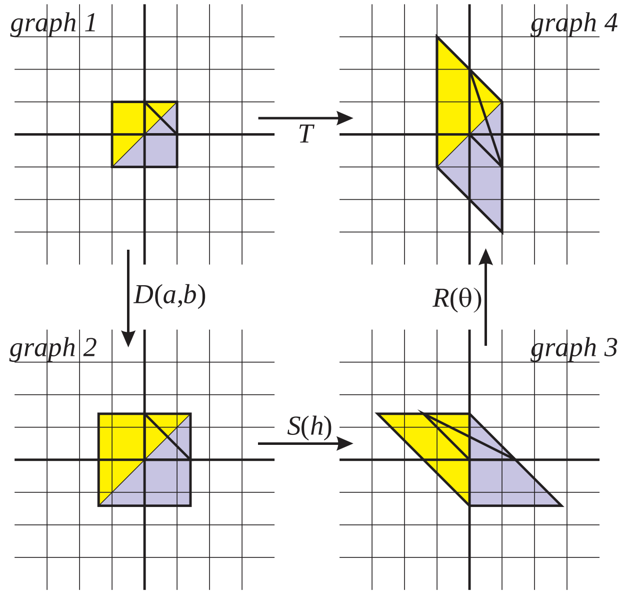
## Example 2 algebra

Find $\theta$, $a$, $b$ and $h$ such that

$$T = R(\theta) \circ S(h) \circ D(a,b)$$

$$[T] = [R(\theta)] \cdot [S(h)] \cdot [D(a,b)]$$

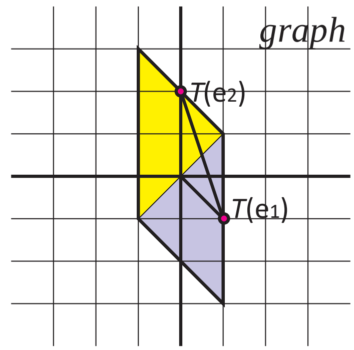
$$\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$




To simplify notation, set $s = \sin\theta$ and $c = \cos\theta$

$$\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \cdot \begin{bmatrix} a & hb \\ 0 & b \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} ca & chb - sb \\ sa & shb + cb \end{bmatrix}$$




$$T(\mathbf{e}_1) = (1,-1)$$
$$T(\mathbf{e}_2) = (0,2)$$

giving us 4 equations in 4 unknowns:

(1) $1 = ca$

(2) $-1 = sa$

(3) $0 = chb - sb$

(4) $2 = shb + cb$

Divide (2) by (1) to get $\frac{-1}{1} = -1 = \frac{sa}{ca} = \frac{s}{c} = \tan\theta$

hence $\theta = \tan^{-1}(-1) = -45°$

From this

$$s = \sin\theta = \sin(-45) = -1/\sqrt{2}$$

$$c = \cos\theta = \cos(-45) = 1/\sqrt{2}$$

From (1) we then get

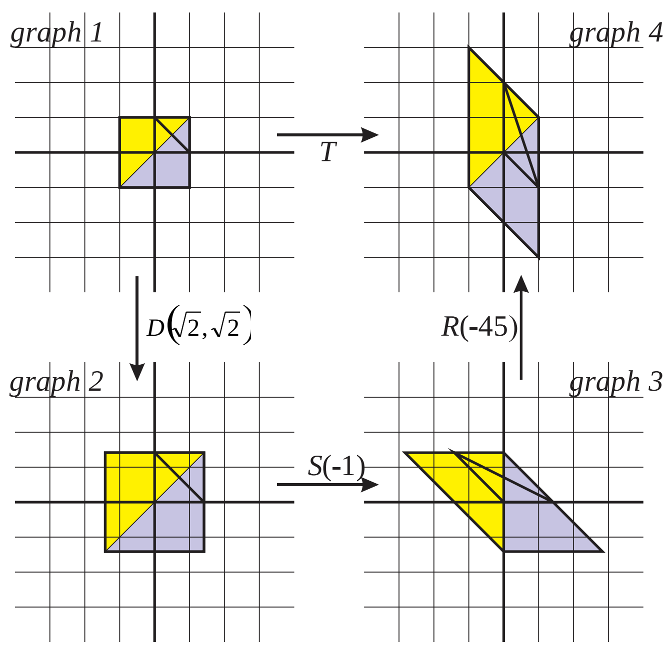
$$a = 1/c = \sqrt{2}\,.$$

From (3): $chb = sb \quad \Rightarrow \quad ch = s$

$$\Rightarrow \quad h = s/c = \tan(-45) = -1$$

From (4): $b = \frac{2}{sh + c} = \frac{2}{(-1/\sqrt{2})(-1) + (1/\sqrt{2})} =$

$$= \frac{2}{(1/\sqrt{2}) + (1/\sqrt{2})} = \frac{2}{2/\sqrt{2}} = \sqrt{2}$$



**Example 2 worksheet**