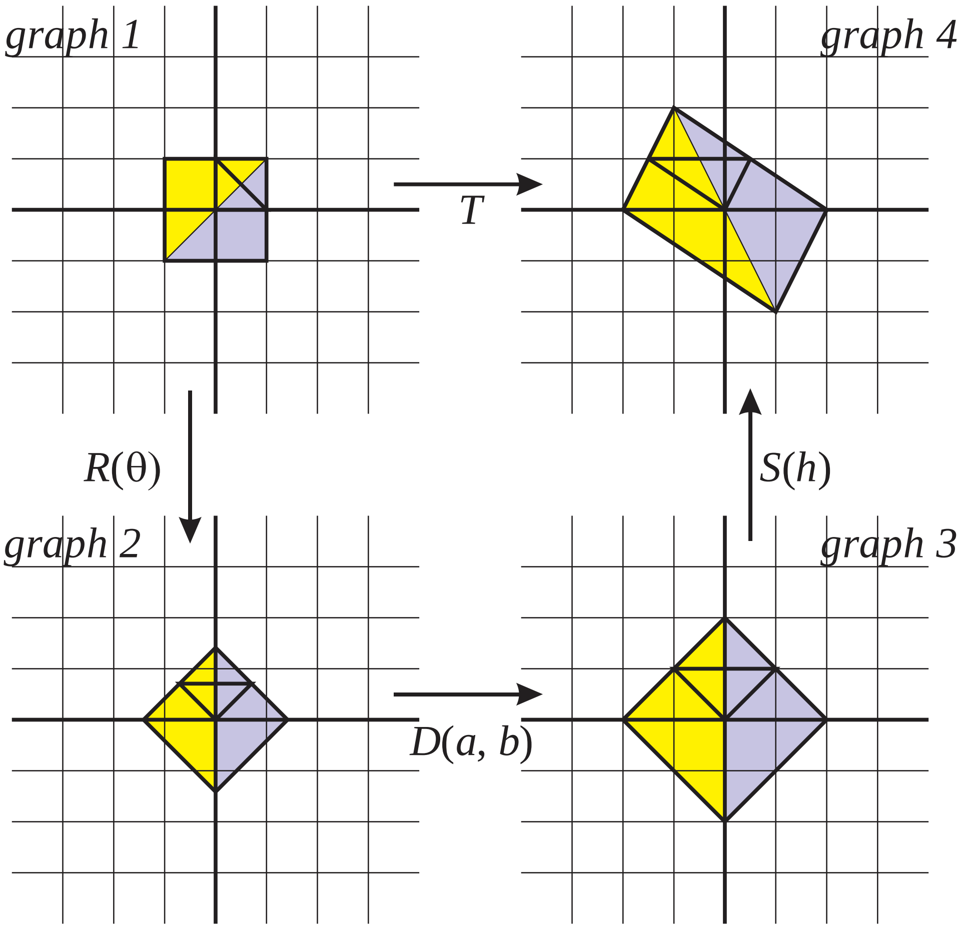
Example 3

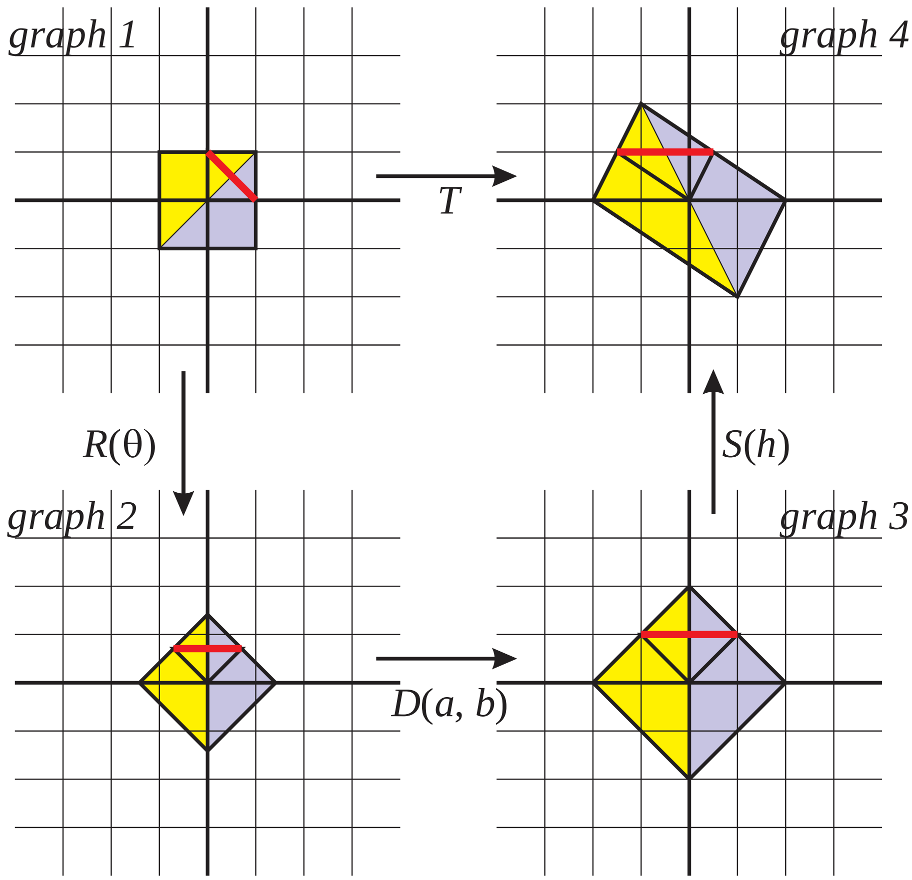
For this example we have reordered the components and put the rotation first, followed by a dilation, then a shear.



Again we ask where we should begin and again we think about invariants. In Examples 1 and 2 we made good use of the fact that dilations and (horizontal) shears both leave horizontal lines horizontal. Does that help us here?

Well it tells us that horizontality is preserved over the *last two steps* of the decomposition above, that is, horizontal lines in graph 2 will still be horizontal in graphs 3 and 4. Can you see how to use that to find the angle of rotation θ?

Finding θ

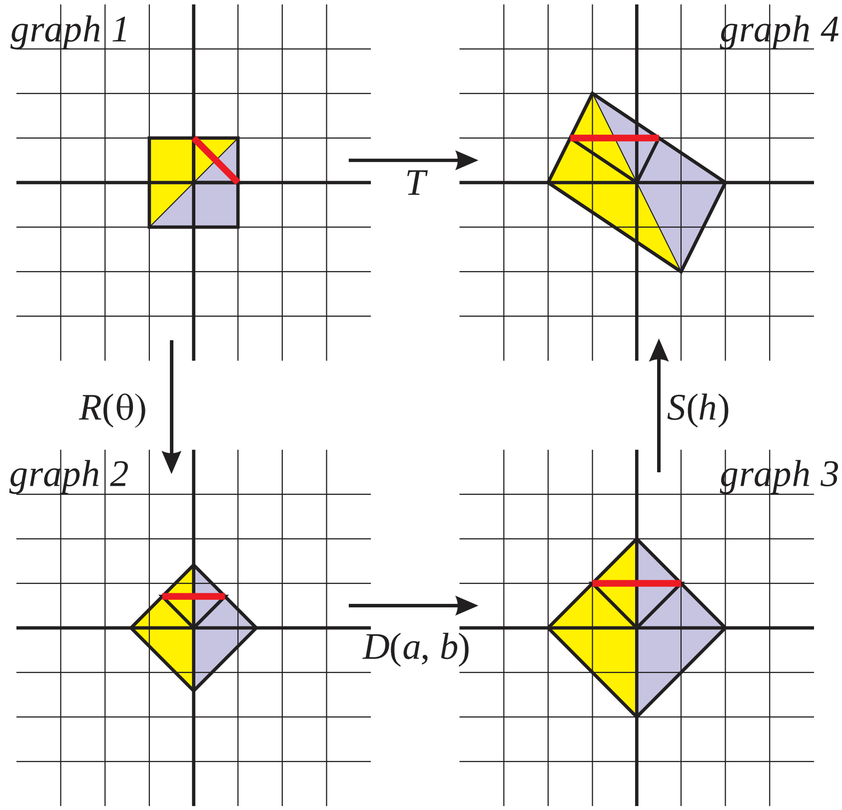


We use the same principle of horizontal invariance we used in Examples 1 and 2 except this time the dilation and shear (that preserve horizontality) are in the last two steps. Thus, if we can find a "simple" (easy to locate) horizontal line in graph 4, we know that its footprint in graph 2 will still be horizontal. Comparing that with its footprint in graph 1 (which won't be horizontal) should allow us to find θ.

A simple such horizontal line in graph 4 is the red line at height $y = 1$. In graph 2 its footprint *must be horizontal* and in graph 1 the grid points tell us that it lies at a 45º angle. Being careful with sign, the correct angle (going from graph 1 to graph 2) is

$$\theta = 45.$$

Finding a

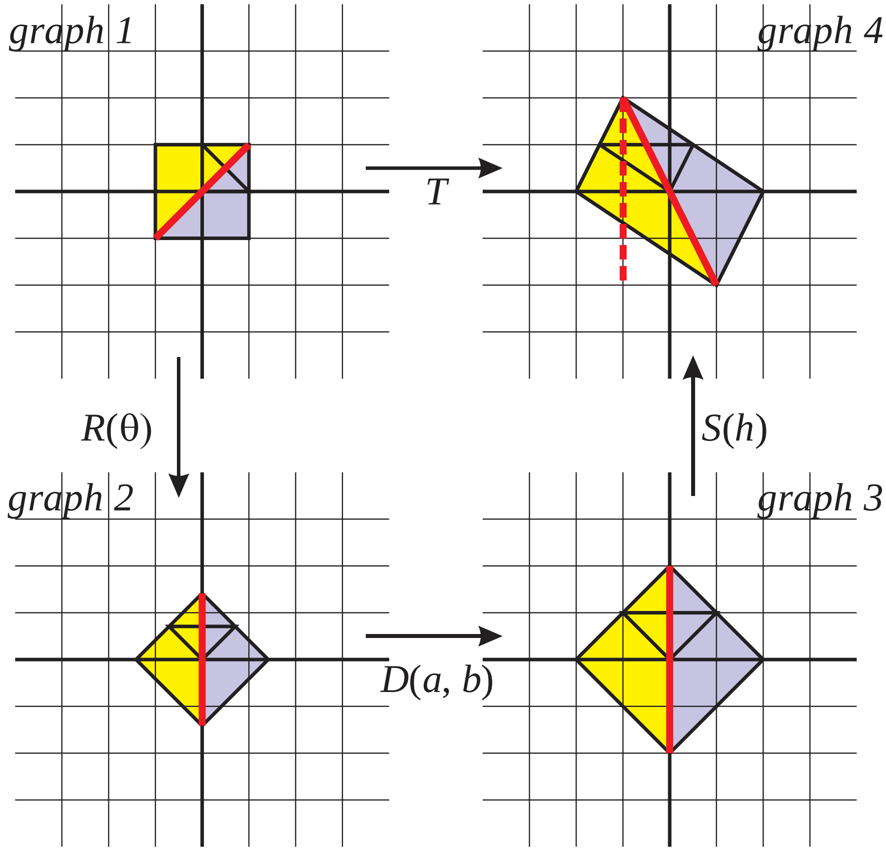


We have to compare the lengths of a horizontal line in graphs 2 and 3. We use the same red line that served us in the θ-analysis. Its length in graph 2 is $\sqrt{2}$ found from its footprint in graph 1 and using the fact that lengths do not change under rotation. Okay, that means that its length in graph 3 is $\sqrt{2}a$. But how long is that?

We need to look at its footprint in graph 4. Will it have the same length? Yes it will!—*because it is horizontal in graph 3 the shear won't change its length*. But using the grid points, its footprint in graph 4 is clearly of length 2. Thus

$$\sqrt{2}a = 2 \quad \Rightarrow \quad a = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

<u>*Finding b*</u>



This time we need to keep track of the length of a vertical line as it jumps from graph 2 to graph 3. An obvious candidate in graph 2 is the boundary between the yellow and grey triangles. It has the same length as its graph-1 footprint which is the diagonal of length $2\sqrt{2}$. Its footprint in graph 3 will then have length $2\sqrt{2}b$.

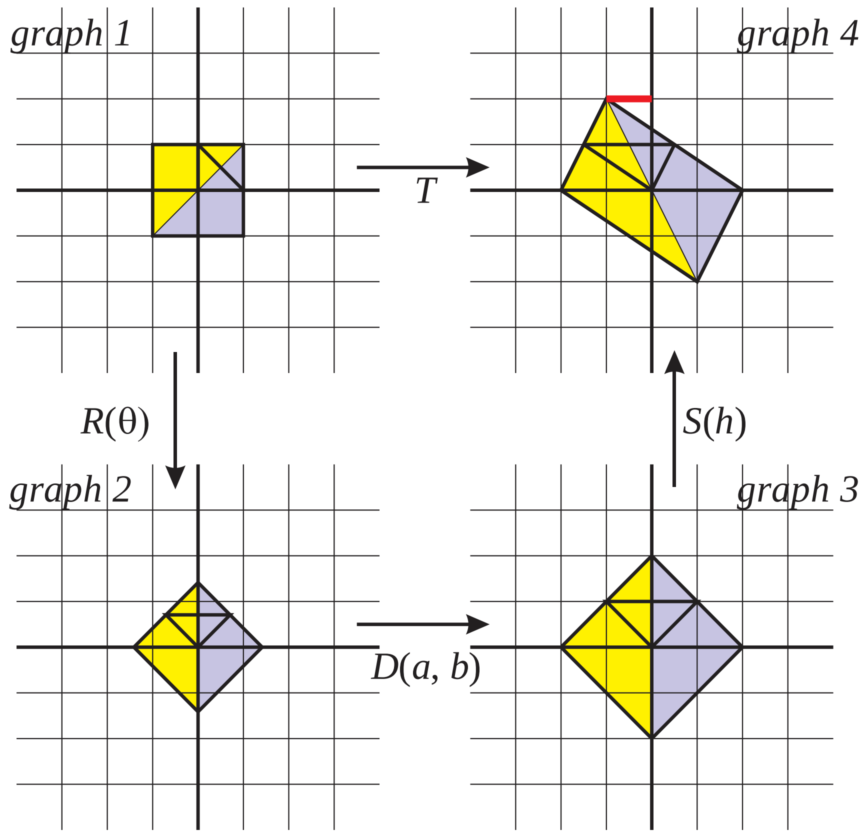
Because of the shear its footprint in graph 4 (solid red) is a bit longer. But shears do not change the vertical distance between points, so that its length will be the vertical distance between the top and bottom vertices of the box in graph 4 (dashed red line), and that's clearly 4.

Thus:

$$2\sqrt{2}b = 4 \quad \Rightarrow \quad b = \frac{4}{2\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

It's interesting that $a = b = \sqrt{2}$ and thus the image in graph 3 is simply magnified by the factor $\sqrt{2}$ and remains a square.

Finding h

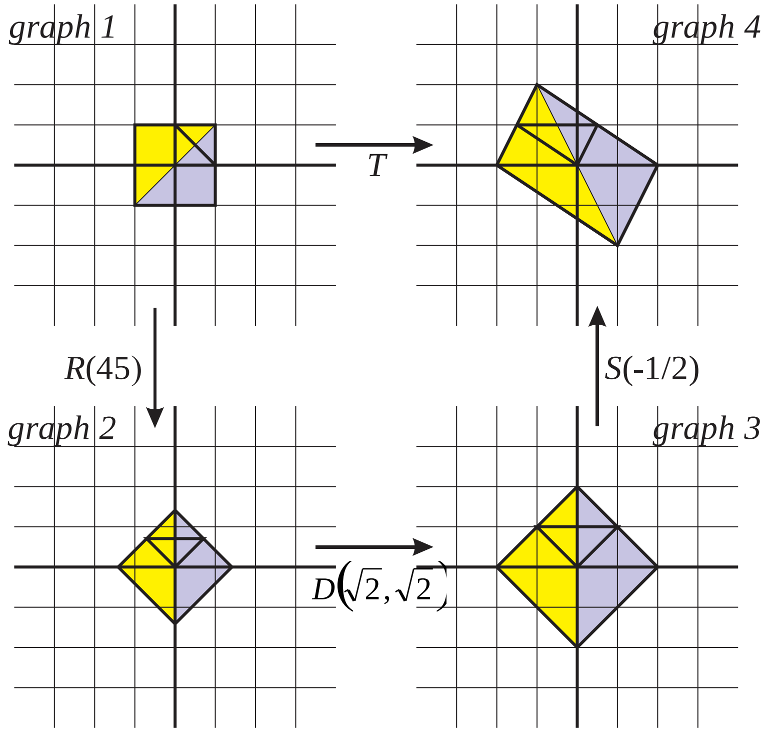


We look for h directly. Consider the top vertex of the box in both graphs 3 and 4. They both have the same height (because the shear only moves points sideways) and from graph 4 that is $y = 2$. What is the horizontal displacement of this vertex between graphs 3 and 4? In graph 3 it is on the y-axis ($x = 0$) and in graph 4 it is at $x = -1$. Thus its displacement is –1. But since it is at height $y = 2$, that displacement is $2h$. That is

$$2h = -1$$

$$h = -1/2.$$

Putting everything together, we have the solution.



By the way, if we interchange the order of the last two blocks, the dilation and the shear, they remain the same as above. That's worth noting as this is not the case in general. Composition of transformations is not "commutative," that is changing the order typically changes the final result. You almost certainly have met this before with ordinary operations on your calculator. For example,

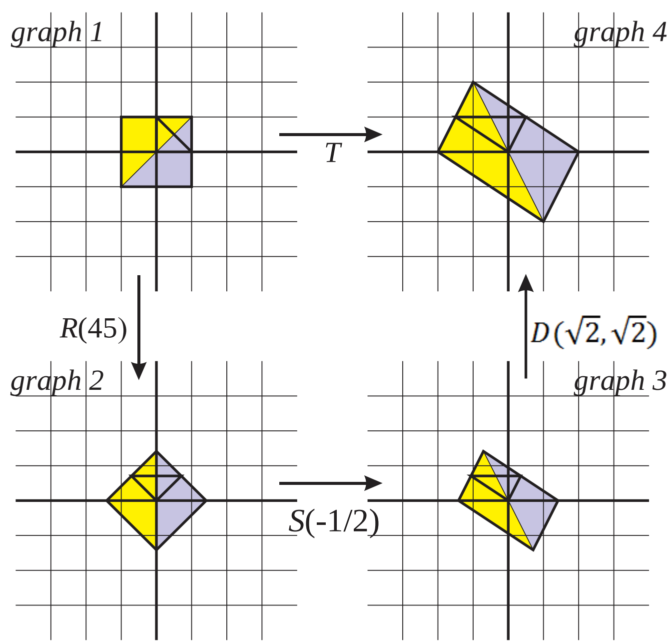
(1) Start with x, square it and then multiply by 2: you get $2x^2$

(2) Start with x, multiply by 2 and then square it: you get $(2x)^2 = 4x^2$

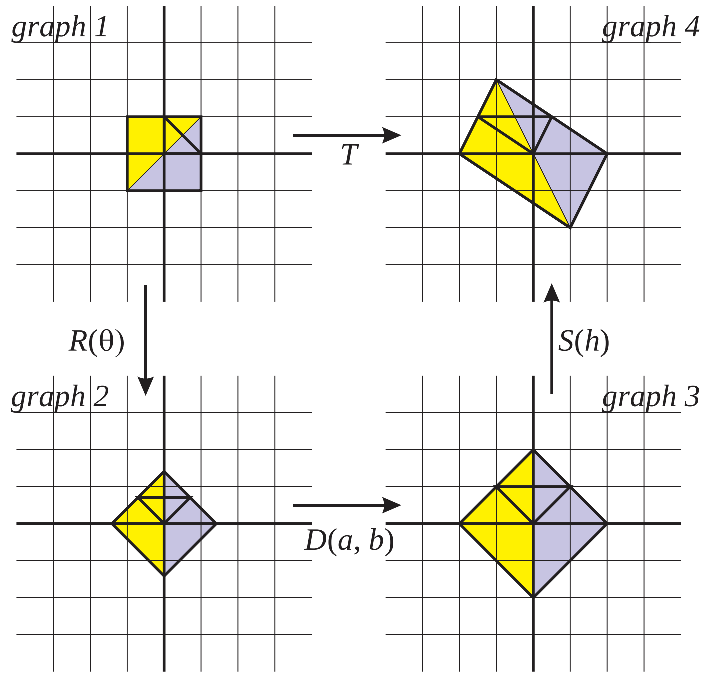
and these are different.

But here the order doesn't matter—you can dilate and then shear, or shear and then dilate—you get the same final result (though the intermediate result is of course different).

The reason this happened is because the two dilation parameters a and b are the same. If this were not the case, we'd get a different result when we interchanged the order.



Example 3 algebra



Find θ, a, b and h such that:

$$T = [S(h)] \cdot [D(a,b)] \cdot [R(\theta)]$$

$$\begin{bmatrix} 1/2 & -3/2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

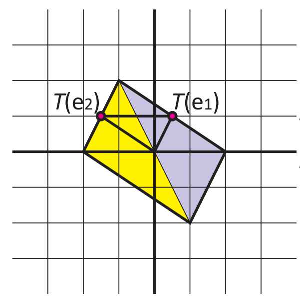
To simplify notation I set s = sinθ and c = cosθ.

$$\begin{bmatrix} 1/2 & -3/2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & hb \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & -3/2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} ac + hbs & -as + hbc \\ bs & bc \end{bmatrix}$$

giving us 4 equations in 4 unknowns:

(1) $ac + hbs = 1/2$
(2) $-as + hbc = -3/2$
(3) $bs = 1$
(4) $bc = 1$



$T(\mathbf{e}_1) = (1/2, 1)$
$T(\mathbf{e}_2) = (-3/2, 1)$

Divide (3) by (4) to get

$$\frac{bs}{bc} = \frac{s}{c} = \tan\theta = \frac{1}{1} = 1$$

Hence θ = 45º

and $s = \sin\theta = 1/\sqrt{2}$

$c = \cos\theta = 1/\sqrt{2}$

From (3) we then get $b = 1/s = \sqrt{2}$

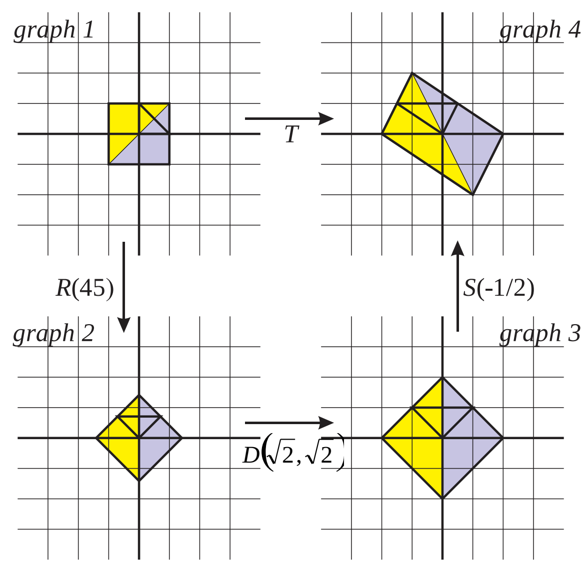
Now using the fact that bs = bc = 1, (1) and (2) can be written:

(1) $ac + h = 1/2 \quad \Rightarrow \quad h = 1/2 - ac$
(2) $-as + h = -3/2 \quad \Rightarrow \quad h = -3/2 + as$

$-3/2 + as = 1/2 - ac$

Thus: $as + ac = 3/2 + 1/2 = 2$

$$a = \frac{2}{s+c} = \frac{2}{1/\sqrt{2} + 1/\sqrt{2}} = \frac{2}{2/\sqrt{2}} = \sqrt{2}$$



From the new (1):

$$h = 1/2 - ac = 1/2 - (\sqrt{2})(1/\sqrt{2}) = 1/2 - 1 = -1/2 \,.$$

Example 3 worksheet