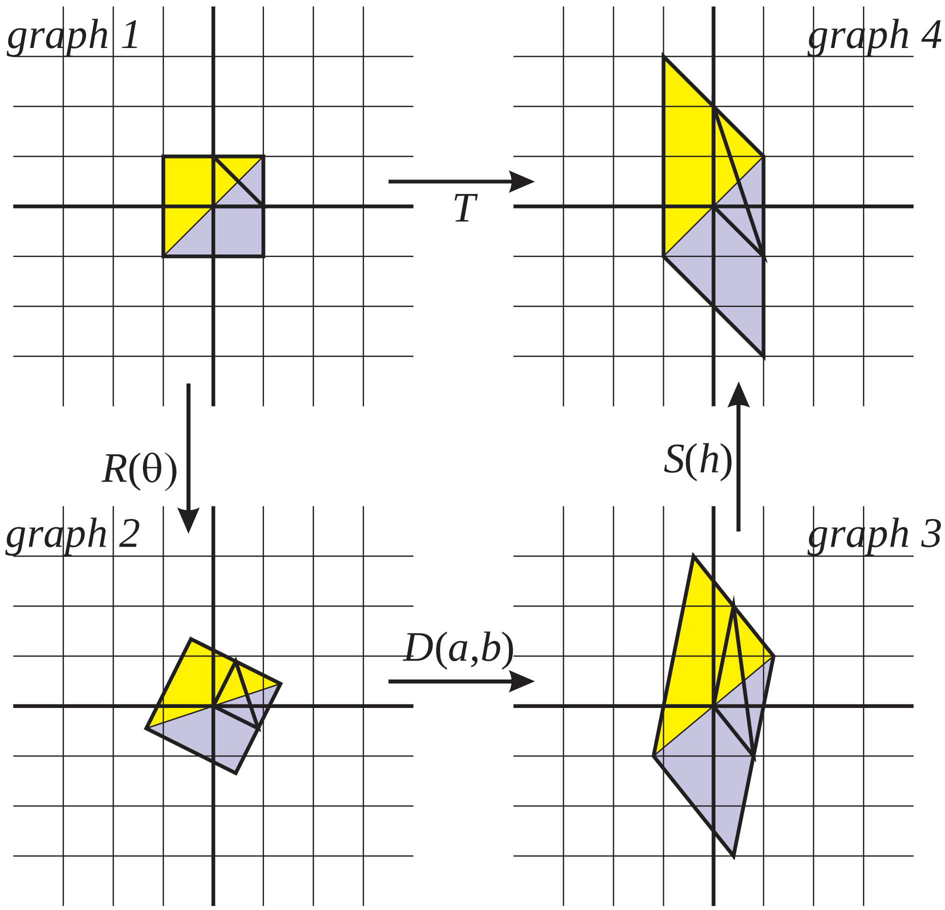
Example 4 geometry.

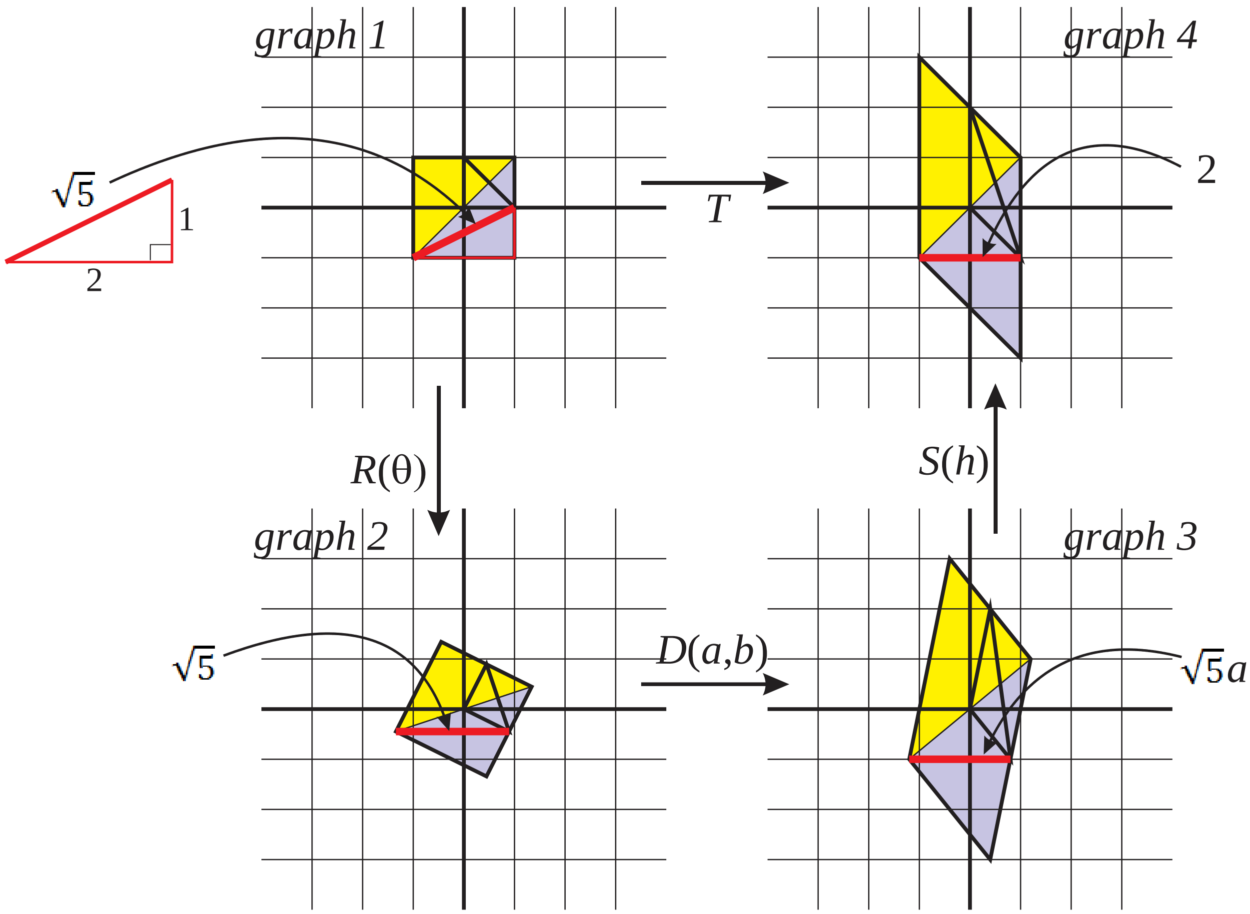
This example is similar to Ex 3, in that the component transformations are in the same order, but it is technically a bit more challenging. For most of the students we have worked with this example marks the end of their journey. Those who have worked more quickly have gone on to do a couple more examples.

Use *geometric* reasoning to find the parameters a. b, h and θ.



Try finding the dilation multiplier a first. For that you will need a pair of horizontal lines in graphs 2 and 3 whose length you can compare. The best strategy is to begin by finding a good horizontal line in graph 4—where you know that the image is anchored to the grid points. You know that such a line will still be horizontal in graphs 3 and 2.

Finding a.



I begin with a, the x-multiplier. We need to find a horizontal line in graph 2 and compare its length with the length of its footprint in graph 3. Using the invariance properties of D and S its footprints in graphs 3 and 4 will also be horizontal. So we look for a horizontal line in graph 4 (where we can make precise measurements) and we find an obvious candidate of length 2. We know that its footprints in graphs 3 and 4 will also be horizontal.

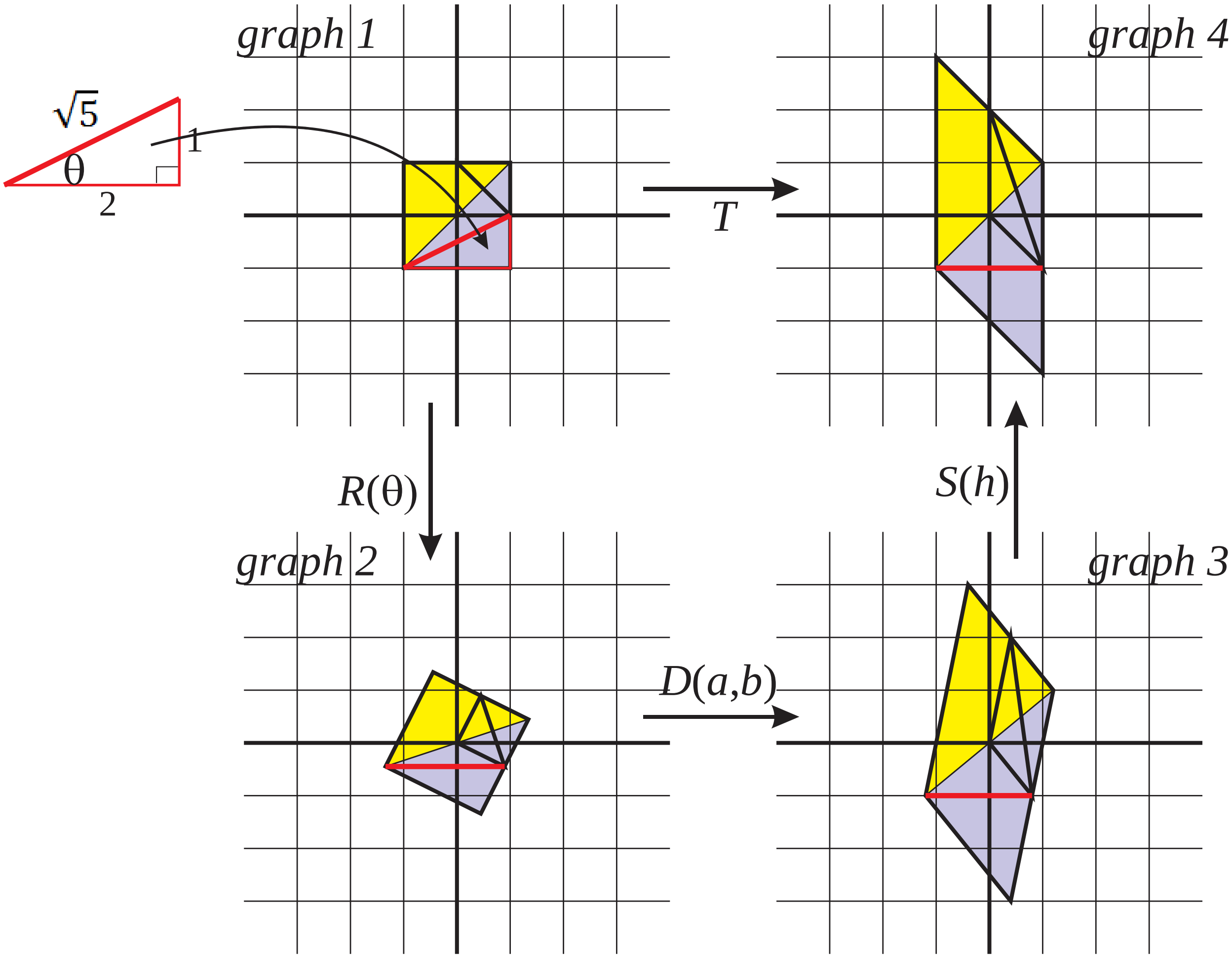
In order to track it through, we first look at its footprint in graph 1 (where we can also make precise measurements) and Pythagoras gives us its length as $\sqrt{1^2+2^2}=\sqrt{5}$.

Now the footprint in graph 2 will also be of length $\sqrt{5}$ (as R does not change the length of lines) and (by definition of a) the graph 3 footprint will then be of length $\sqrt{5}a$.

Finally since S does not change the length of horizontal lines, the line has the same length in graphs 3 and 4 and hence $\sqrt{5}a = 2$ so that

$$a = 2/\sqrt{5}\,.$$

Finding θ.

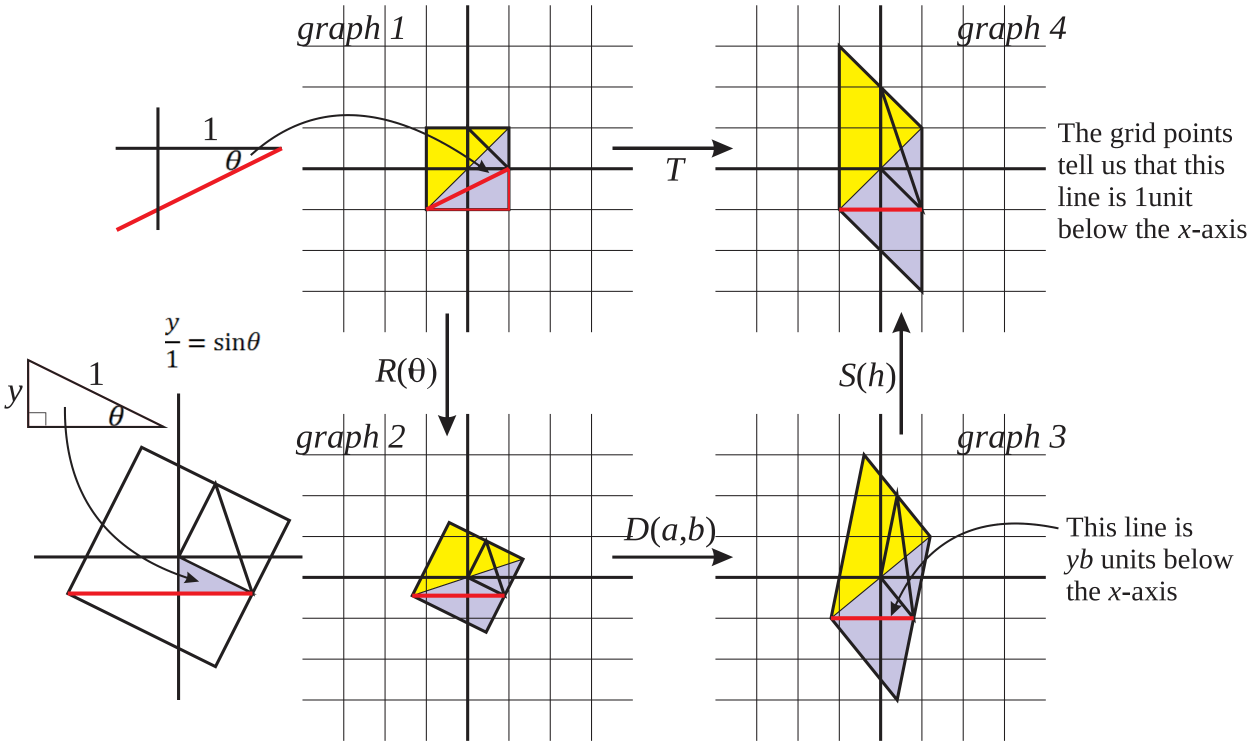


For this we use the same line we used above. It is horizontal in graph 2 and to become that from graph 1 it has had to rotate *clockwise* through an angle θ whose tangent is ½. That makes the angle negative, so that

$\theta = -\tan^{-1}(1/2) = -26.57°$.

Finding b.

The parameter b is the y-multiplier for the dilation so we have to follow the change in a *vertical* distance between graphs 2 and 3.



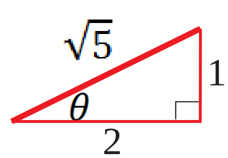
I will continue to use the same horizontal red line we have been using for a and θ, but this time I will keep track, not of its length but of its distance below the x-axis.

Let's start in graph 2 and let the line be y units below the x-axis. We see that y is the side-length of a right-angled triangle with opposite angle θ and hypotenuse 1. To see that, we identify those segments in graph 1 where we can make precise measurements. From the graph 2 triangle,

$$\frac{y}{1} = \sin\theta$$

But since $\tan\theta = ½$, we have $\sin\theta = \frac{1}{\sqrt{5}}$ and hence

$$y = 1/\sqrt{5}.$$

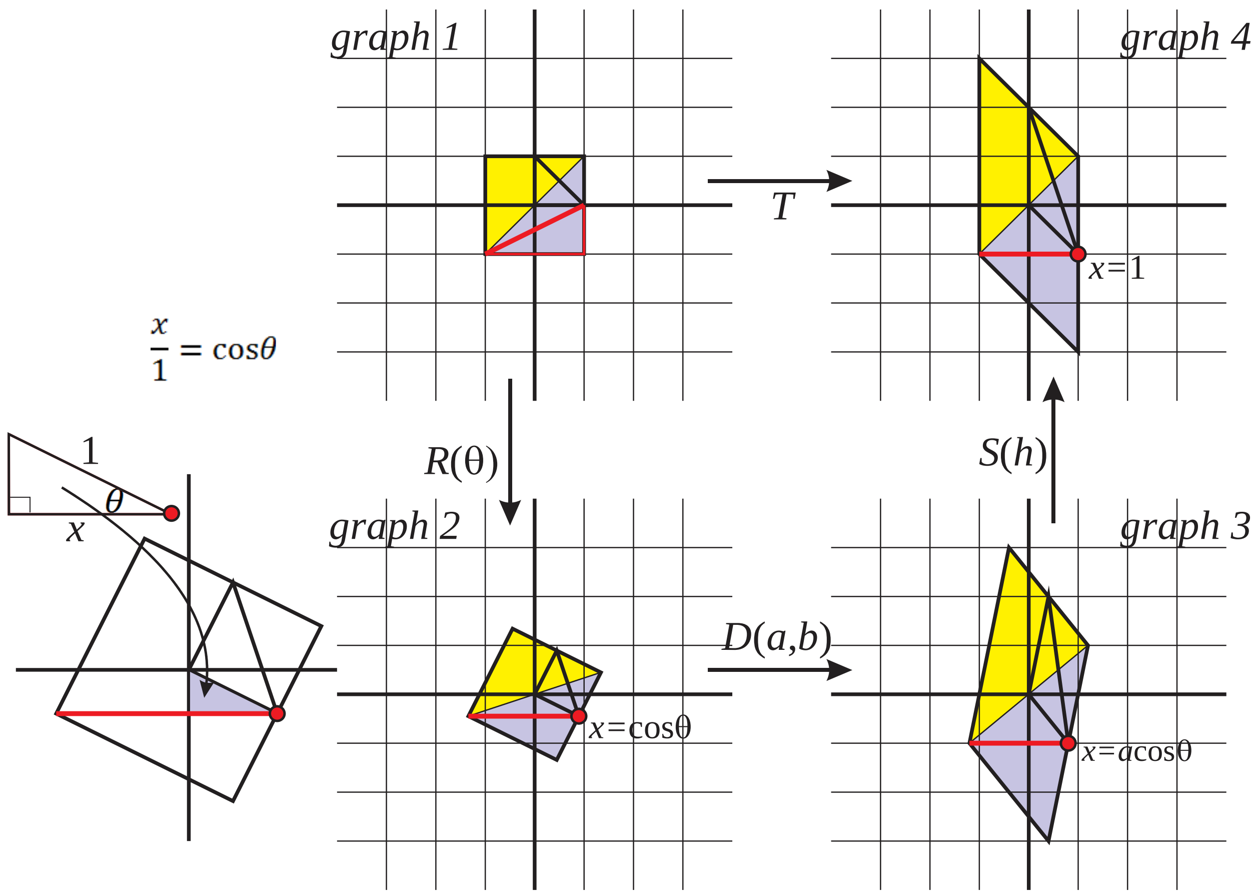


Now by definition of b, the footprint of the line in graph 3 is by units below the x-axis. But in graph 4 where we can make precise measurements it is 1 unit below the x-axis. Since S does not change distances between horizontal lines, by must be the same as 1:

$$by = 1$$

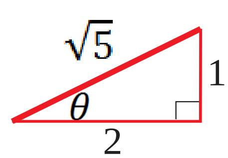
$$b = 1/y = \sqrt{5}.$$

Finding h.



We choose and mark a point that has height –1 (y-coordinate) in graphs 3 and 4. Then $-h$ will be the distance the point moves sideways from graph 3 to graph 4. Now in graph 4 the point has x-coordinate 1, so we need to find its x-coordinate in graph 3. Going back to graph 2, we can get this from the small triangle we used above to find b. This time we have $\cos\theta = x/1$ and thus in graph 2:

$$x = \cos\theta = \frac{2}{\sqrt{5}}$$



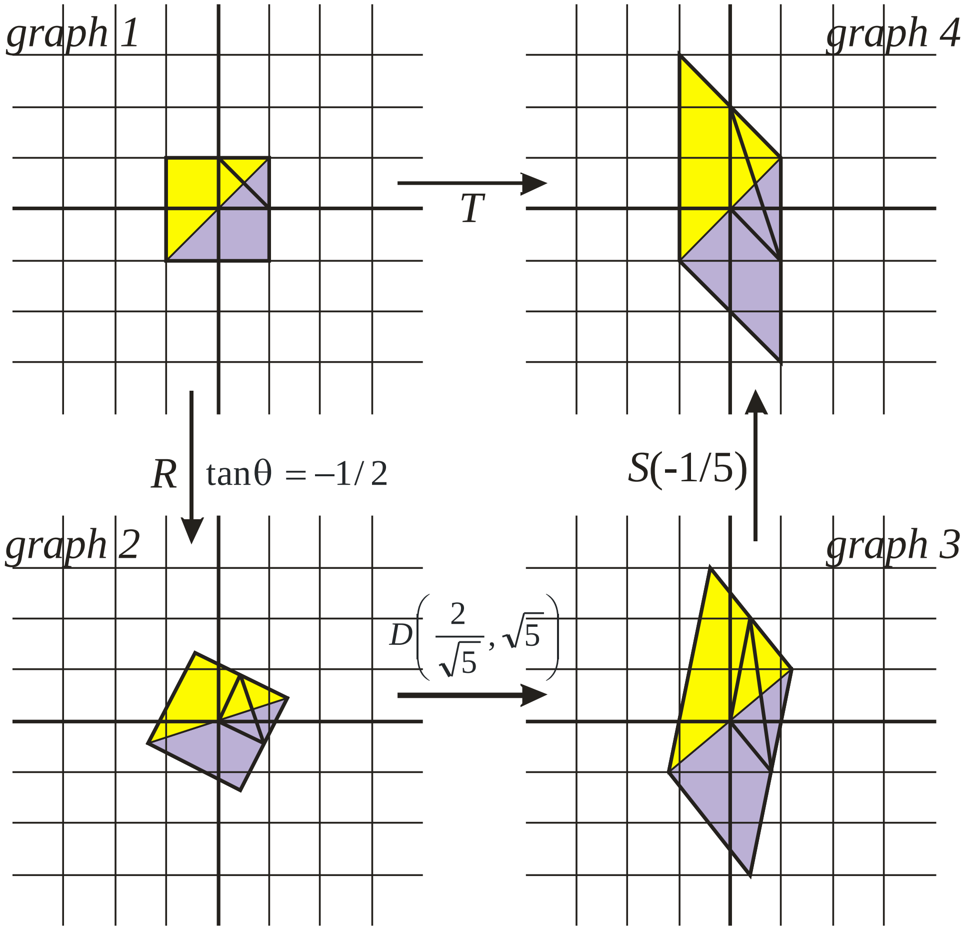
and thus in graph 3:

$$x = a\cos\theta = \left(\frac{2}{\sqrt{5}}\right)\left(\frac{2}{\sqrt{5}}\right) = \frac{4}{5}$$

The horizontal shift required to change x = 4/5 (graph 3) into x = 1 (graph 4) is 1/5. Since these points are at height –1, this gives us:

$$h = -\frac{1}{5}$$

The final solution is illustrated below.



Exercise: What happens if we interchange *D* and *S*—that is, *R* followed by *S*, followed by *D*?

Example 4 algebra

Find θ, a, b and h such that:

$$T = [S(h)] \cdot [D(a,b)] \cdot [R(\theta)]$$

$$\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

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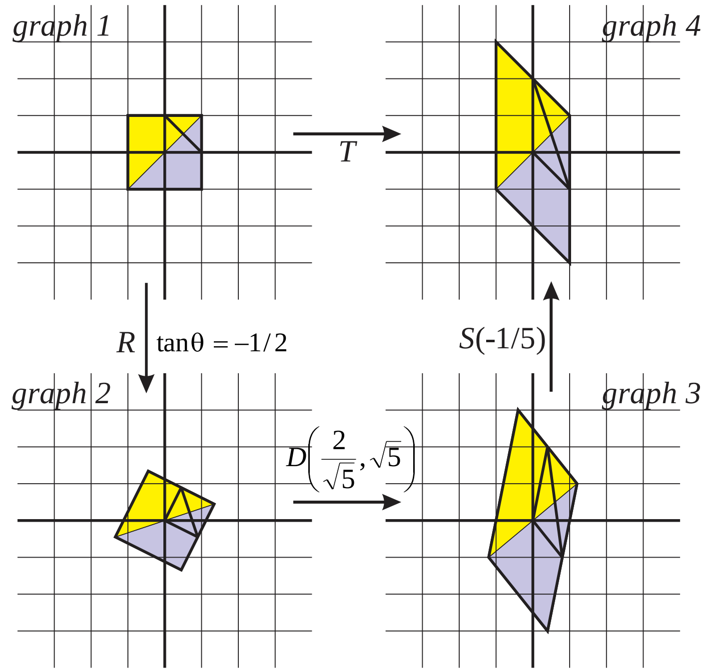


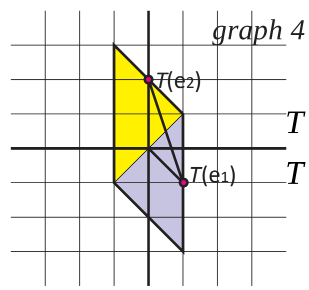
To simplify notation I set $s = \sin\theta$ and $c = \cos\theta$.

$$\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} a & hb \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} ac + hbs & -as + hbc \\ bs & bc \end{bmatrix}$$

giving us 4 equations in 4 unknowns:

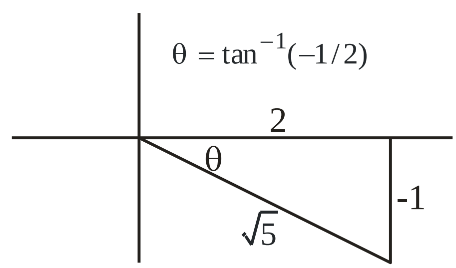
(1) $ac + hbs = 1$
(2) $-as + hbc = 0$
(3) $bs = -1$
(4) $bc = 2$



$T(\mathbf{e}_1) = (1,-1)$
$T(\mathbf{e}_2) = (0,2)$

Divide (3) by (4) to get $\tan\theta = \dfrac{s}{c} = -\dfrac{1}{2}$

It is clear from the diagram that θ is a negative angle in quadrant 4.



hence

$$s = \sin\theta = -1/\sqrt{5}$$

$$c = \cos\theta = 2/\sqrt{5}$$

From (3) we then get $b = \sqrt{5}$

From (1) and (3) $\dfrac{2a}{\sqrt{5}} - h = 1$ $\quad h = \dfrac{2a}{\sqrt{5}} - 1$

From (2) and (4) $\dfrac{a}{\sqrt{5}} + 2h = 0$ $\quad h = -\dfrac{a}{2\sqrt{5}}$

$$\Rightarrow \quad \frac{2a}{\sqrt{5}} - 1 = -\frac{a}{2\sqrt{5}} \quad \Rightarrow \quad a = \frac{2}{\sqrt{5}}.$$

Putting this into the equation $h = -\dfrac{a}{2\sqrt{5}}$ gives us $h = -\dfrac{2}{\sqrt{5}}\dfrac{1}{2\sqrt{5}} = -\dfrac{1}{\sqrt{5}\cdot\sqrt{5}} = -\dfrac{1}{5}$.

Example 4 worksheet