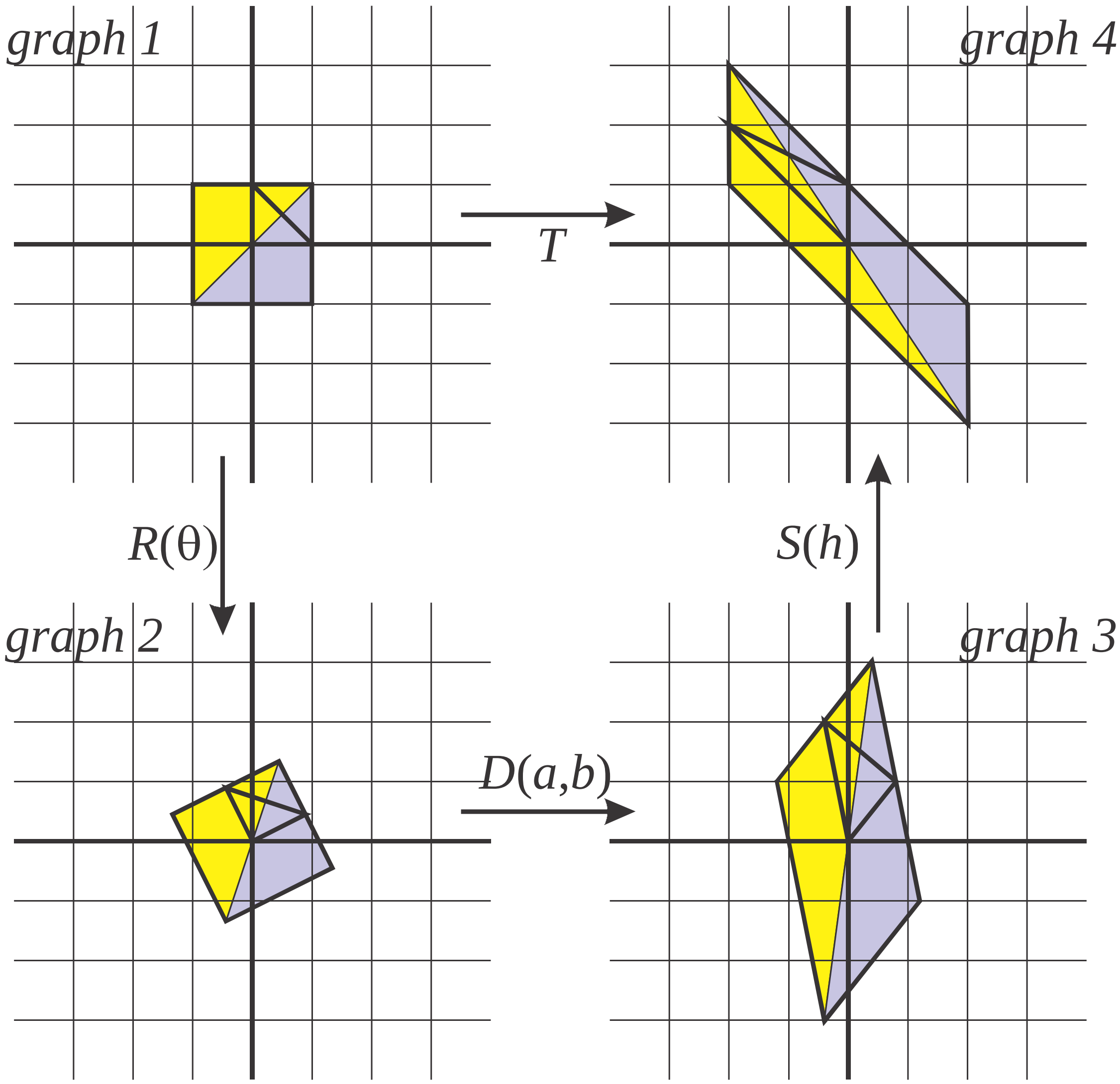
Example 5—geometry

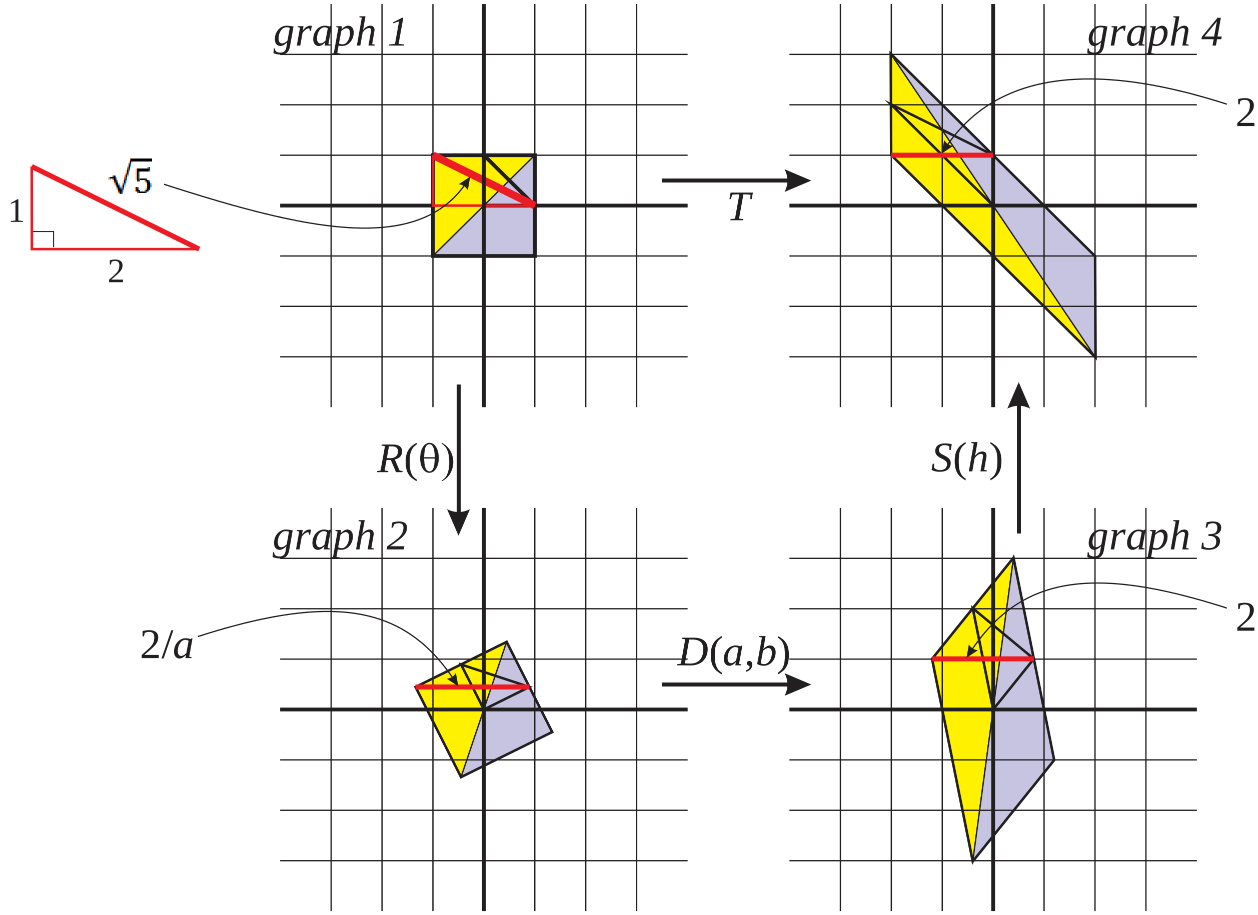
Use *geometric* reasoning to find the parameters a. b, h and θ.



Try finding the dilation multiplier a first. For that you will need a pair of horizontal lines in graphs 2 and 3 whose length you can compare. The best strategy is to begin by finding a good horizontal line in graph 4—where you know that the image is anchored to the grid points. You know that such a line will still be horizontal in graphs 3 and 2.

Solution.

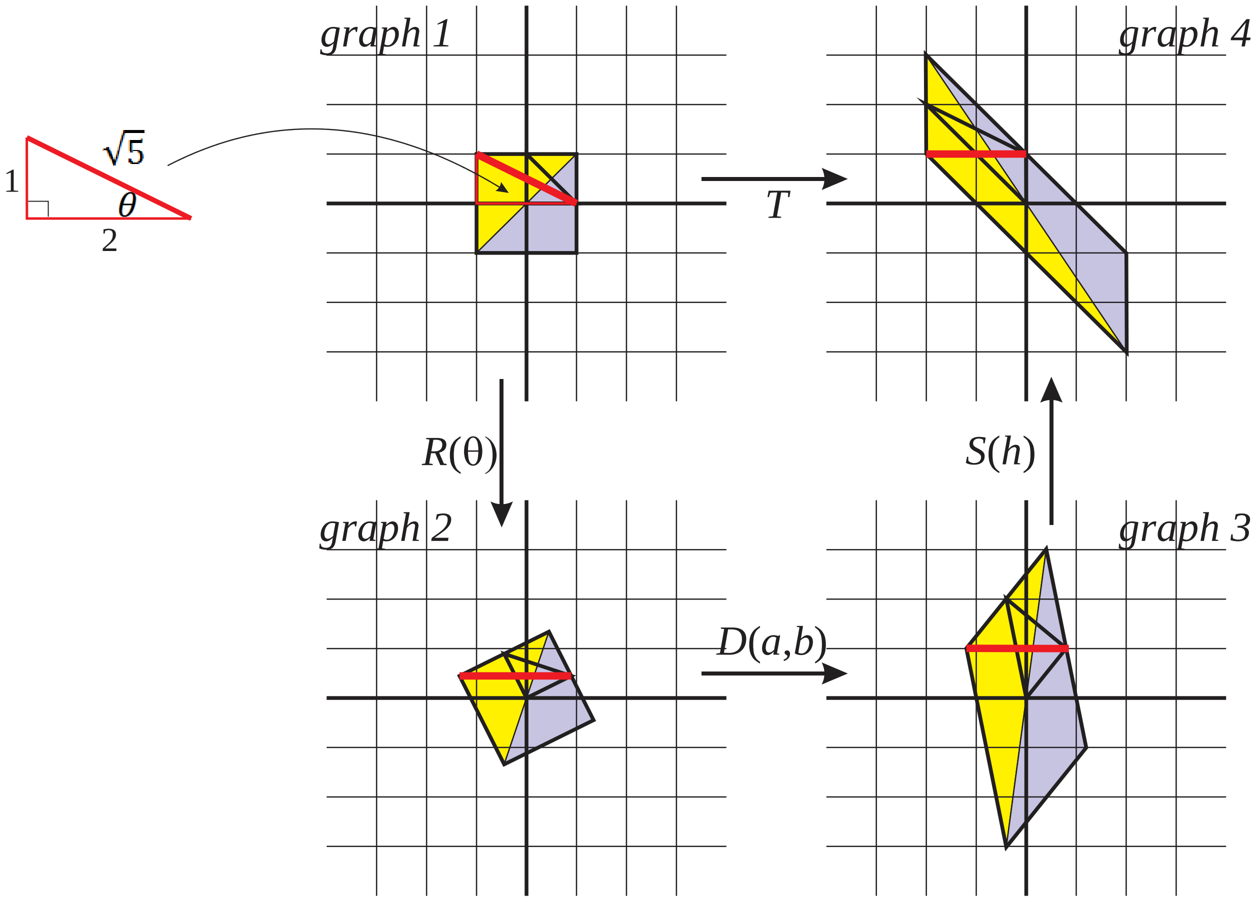
Finding *a*.



Let's start with *a*, the horizontal multiplier. We need to find a horizontal line in graph 2 and compare its length with the length of its footprint in graph 3. Of course its footprint in graph 3 will be horizontal as well and in fact so will the footprint in graph 4 as *S* only moves horizontal lines sideways. So we look for a horizontal line in graph 4 and we find an obvious candidate of length 2.

Okay. Since *S* does not change the length of horizontal lines, the graph 3 footprint will also be of length 2 and then (by definition of *a*) the graph 2 footprint will be of length 2/*a*. But since *R* preserves the length of lines the footprint in graph 1 will have the same length as in graph 2 and Pythagoras tells us that this is $\sqrt{1^2+2^2}=\sqrt{5}$. Hence $2/a=\sqrt{5}$ so that $=2/\sqrt{5}$.

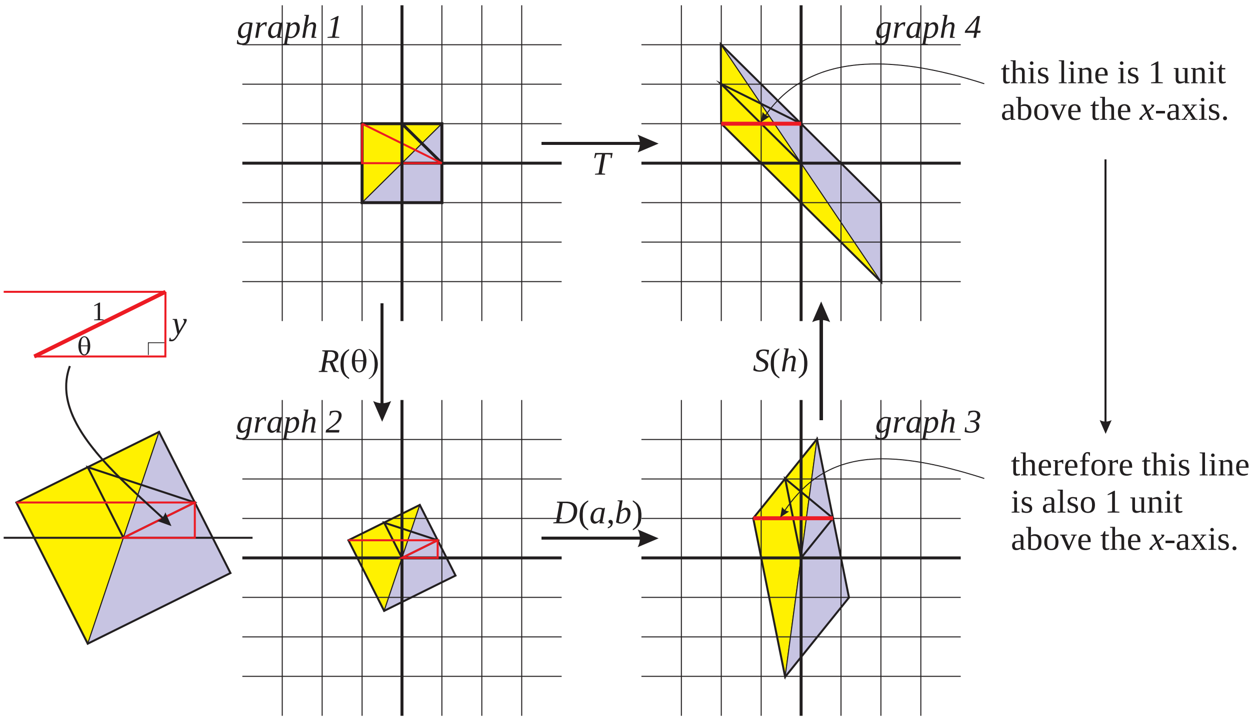
Finding θ.



For this we can use the same line we used above. It is horizontal in graph 2 and to get to graph 1 it has turned counterclockwise through an angle θ whose tangent is ½. Then to go from graph 1 to graph 2 we rotate it counter-clockwise by

$\theta = \tan^{-1}(1/2) = 26.57°$.

Finding b.

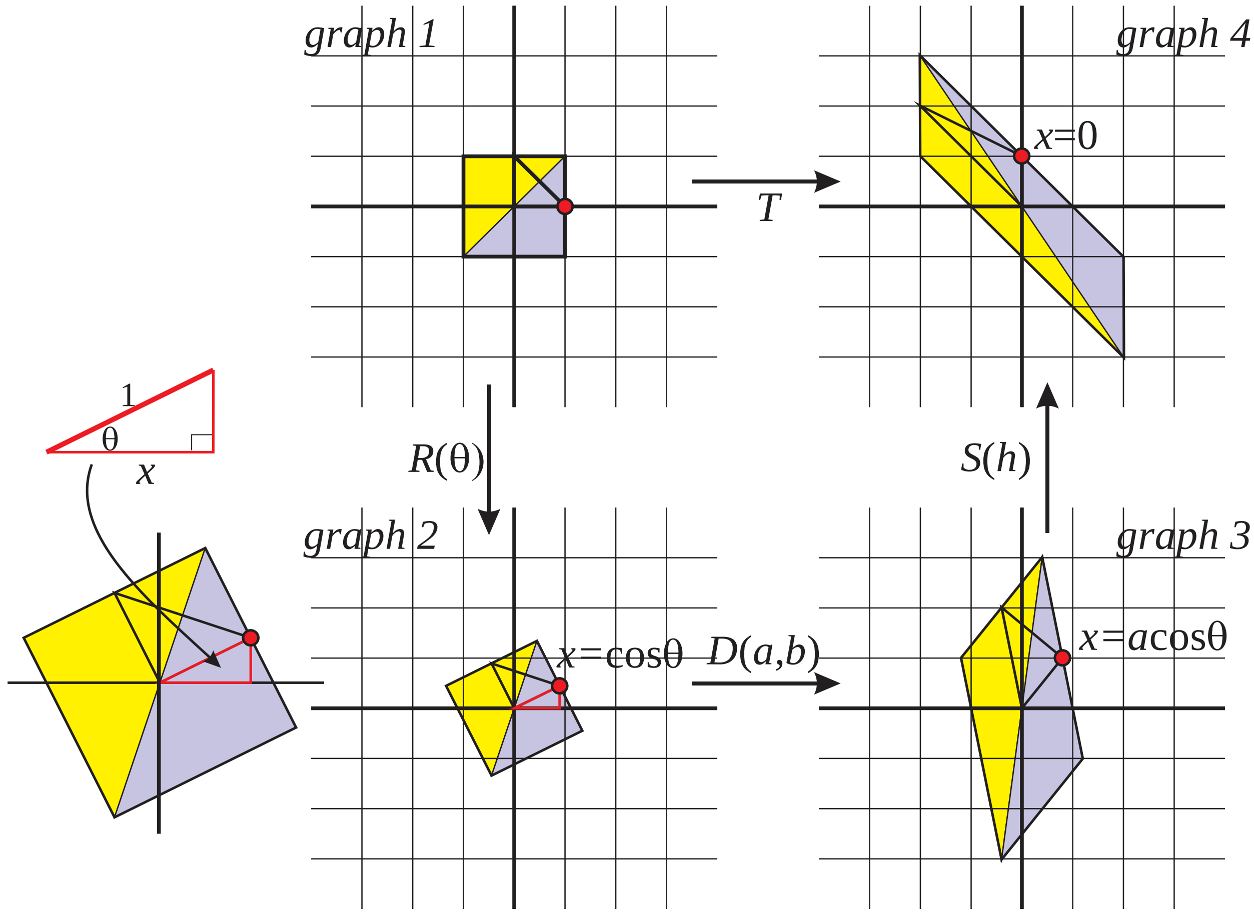


We fasten attention on that same horizontal line, but this time we look at its distance above the x-axis. In graph 3 it is 1 unit above the x-axis. [To be sure of that we need to start with graph 4 where we know we are anchored at the grid points. Then we use the fact that S preserves distance between horizontal lines.]

Let the footprint of that line in graph 2 be y units above the x-axis. If we look in the right place we can find a triangle in graph 2 with a side of length y. Our angle θ is one of the angles of that triangle and its hypotenuse is of length 1 (we can see that in graph 1). From that triangle, $\frac{y}{1} = \sin\theta = \frac{1}{\sqrt{5}}$ and this gives us $y = 1/\sqrt{5}$. Now by definition of b, $by = 1$, and hence

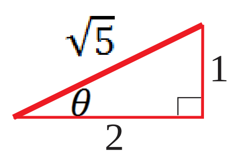
$b = 1/y = \sqrt{5}$.

Finding *h*.



We choose and mark a point that has height 1 (*y*-coordinate) in graphs 3 and 4. Then *h* will be the distance the point moves sideways from graph 3 to graph 4. Now in graph 4 the point has *x*-coordinate 0, so we need to find its *x*-coordinate in graph 3. Going back to graph 2, we can get this from the small triangle we used above to find *b*. We have cosθ =*x*/1 and thus in graph 2:

$x = \cos\theta = 2/\sqrt{5}$



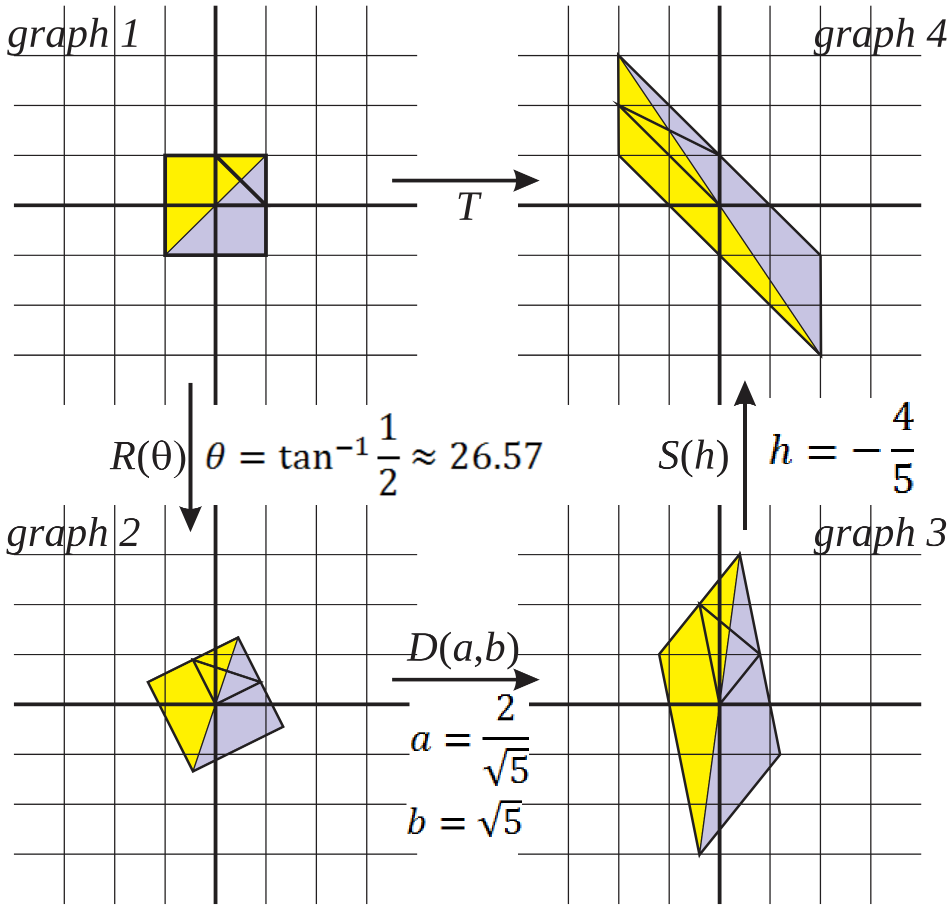
And thus in graph 3:

$x = a\cos\theta = \left(\frac{2}{\sqrt{5}}\right)\left(\frac{2}{\sqrt{5}}\right) = \frac{4}{5}$

The horizontal shift required to change *x* = 4/5 (graph 3) into *x* = 0 (graph 4) is –4/5. Since these points are at height 1, this gives us *h*:

h = –4/5.

Here is the final result

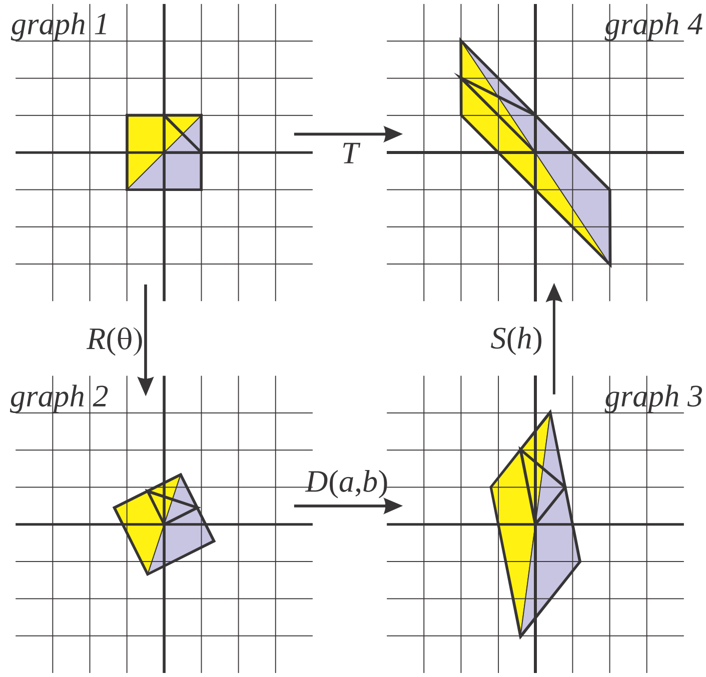


Example 5—algebra

Find θ, a, b and h such that:

$$T = [S(h)] \cdot [D(a,b)] \cdot [R(\theta)]$$

$$\begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



Set $s = \sin\theta$ and $c = \cos\theta$ and multiply the first two matrices on the left.

$$\begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} a & hb \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} ac + hbs & -as + hbc \\ bs & bc \end{bmatrix}$$

giving us 4 equations in 4 unknowns:

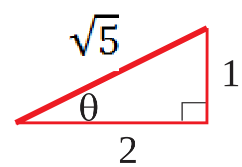
(1) $ac + hbs = 0$

(2) $-as + hbc = -2$

(3) $bs = 1$

(4) $bc = 2$

Divide (3) by (4) to get $\frac{bs}{bc} = \frac{s}{c} = \tan\theta = \frac{1}{2}$



Hence $\theta = \tan^{-1}(1/2) = 26.57^{\circ}$ and $s = \sin\theta = 1/\sqrt{5}$

$c = \cos\theta = 2/\sqrt{5}$

Then from (3) $b = \frac{1}{s} = \sqrt{5}$

That leaves us with eqs (1) and 92) and there are many ways to unwind them. One nice way is to solve both of them for hb. We get:

(1) $hb = -\frac{ac}{s} = -2a$

(2) $hb = \frac{as-2}{c} = a\frac{s}{c} - \frac{2}{c} = a\frac{1}{2} - \sqrt{5}$

Equating the left-hand sides:

$$\frac{a}{2} - \sqrt{5} = -2a$$

$$2a + \frac{a}{2} = \sqrt{5} \implies \frac{5a}{2} = \sqrt{5} \implies a = \frac{2\sqrt{5}}{5} = 2/\sqrt{5}$$

Finally from (1):

$$h = -\frac{2a}{b} = -2a \cdot \frac{1}{b} = -\frac{4}{\sqrt{5}}\frac{1}{\sqrt{5}} = -\frac{4}{5}$$

Example 5 worksheet