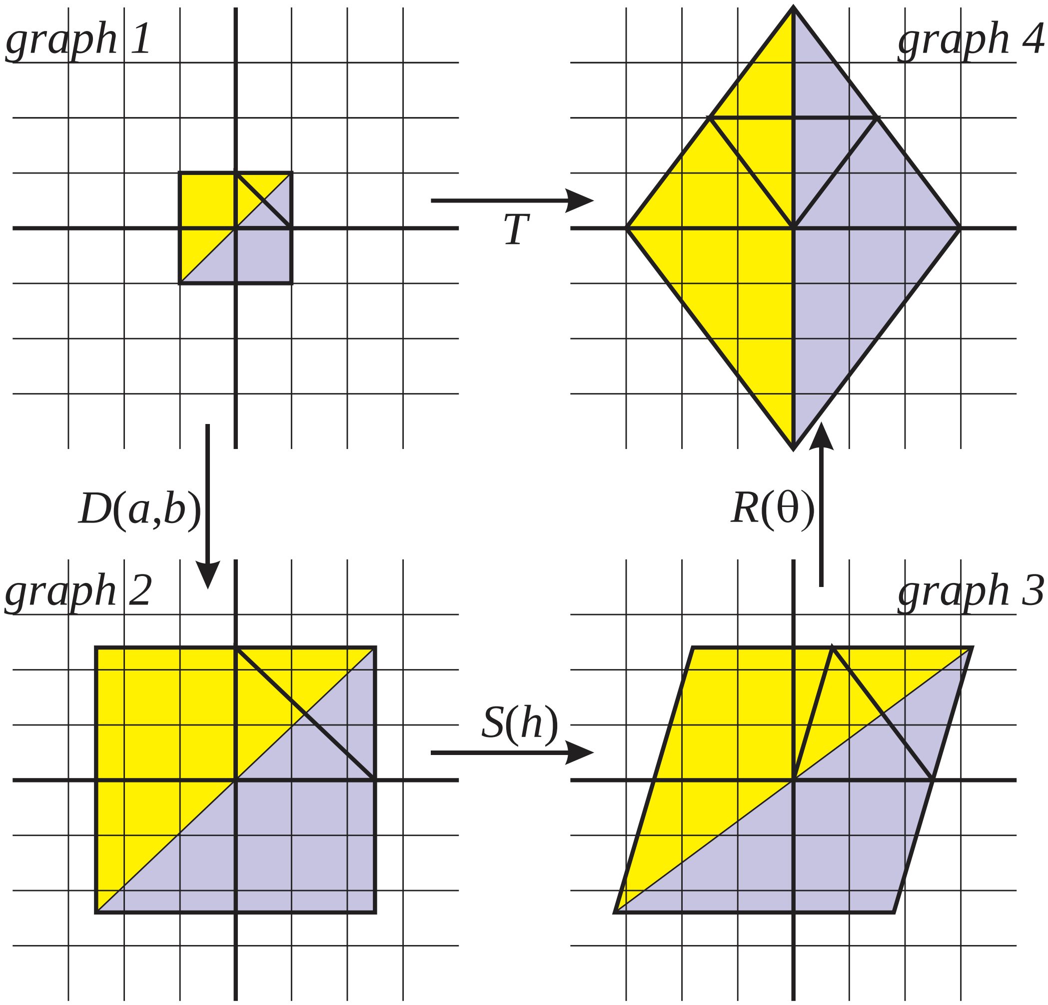
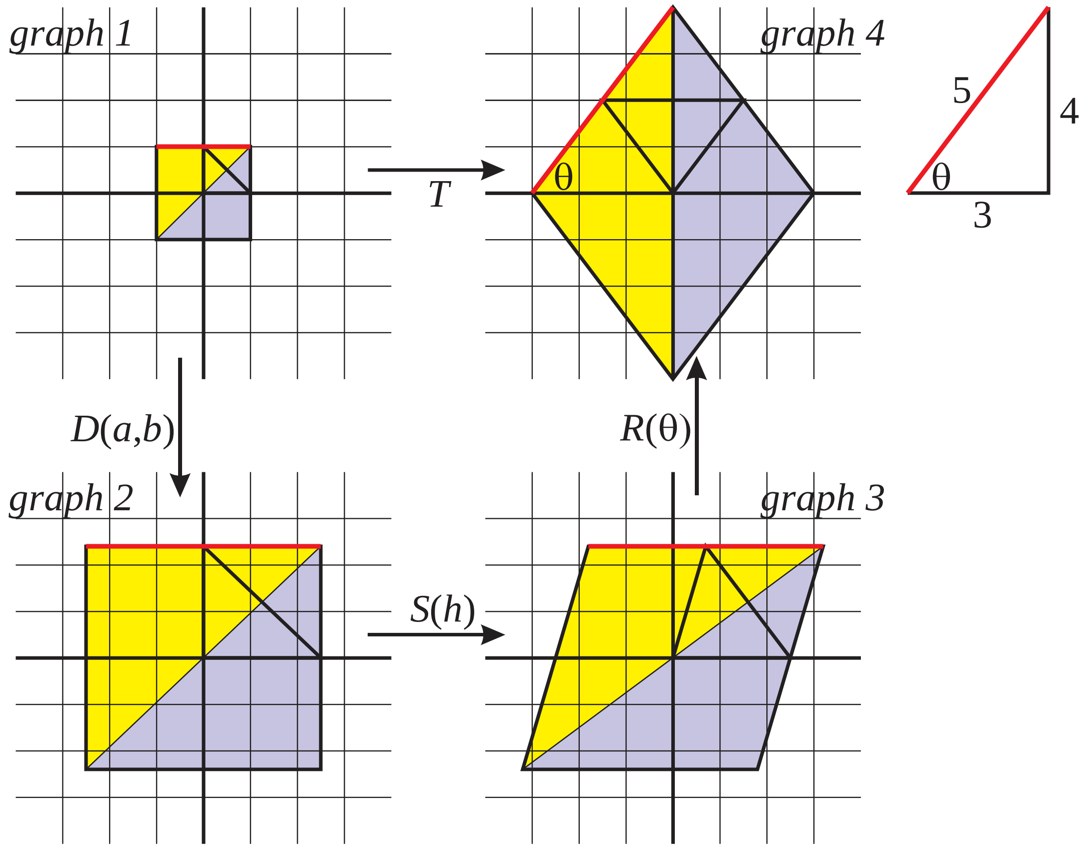
Example 6 geometry

Using geometric reasoning, find the parameters *a*, *b*, *h* and *θ*.



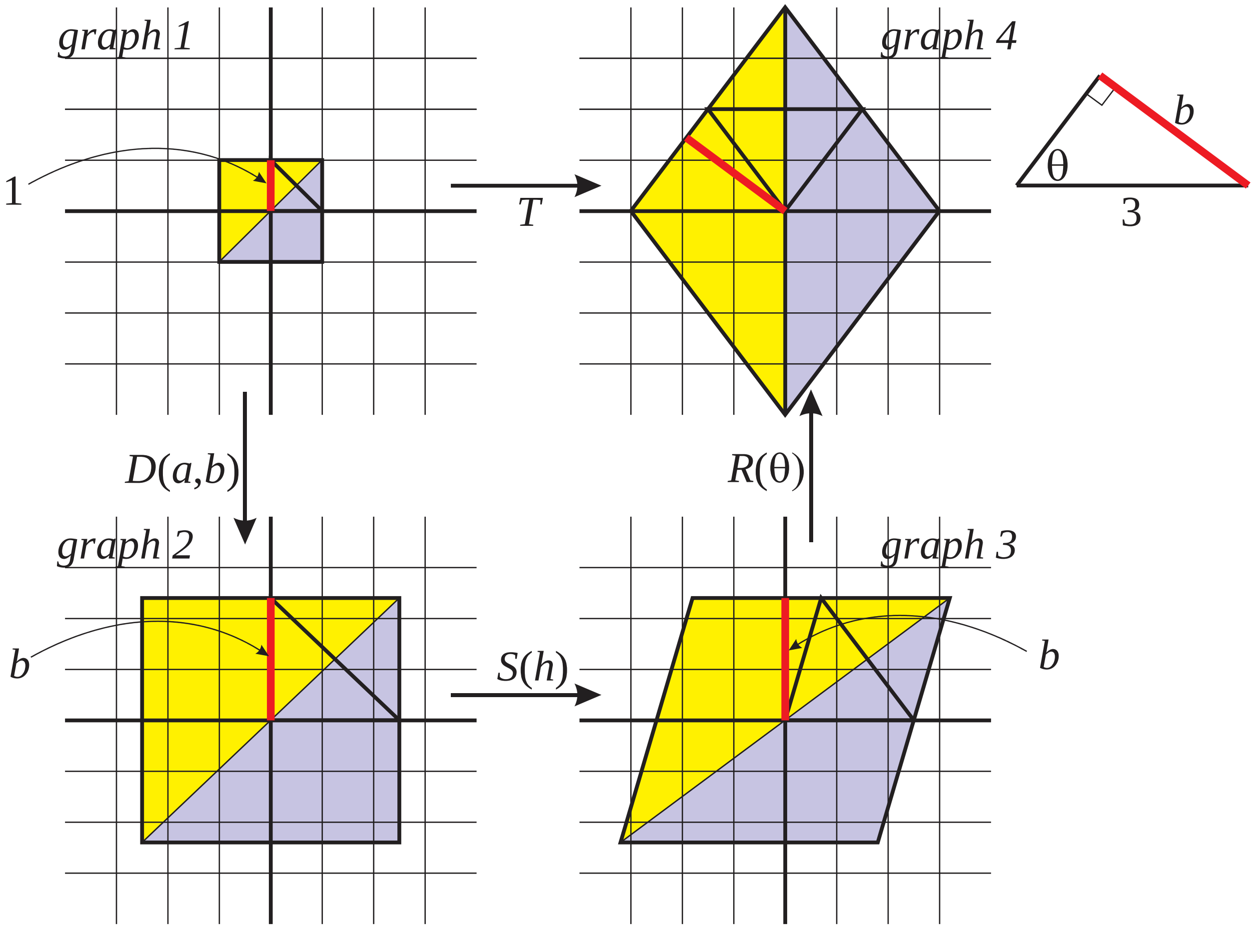
Finding θ and a



The argument for θ is quite familiar. The top of the box remains horizontal in graphs 2 and 3 (because *D* and *S* preserve this property) and then in the passage to graph 4 it turns through the angle θ whose tangent is 4/3. It's a positive angle (counter-clockwise) so $\theta = \tan^{-1}(\frac{4}{3})$ = 53.13°.

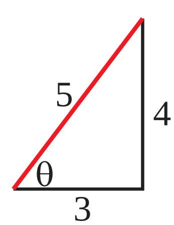
The argument for *a*, the *x*-multiplier of *D*, is also familiar. The top of the graph-1 box has length 2. Its footprint in graph 2 is then of length 2*a* and since it is horizontal, the graph-3 footprint is also of length 2*a*. This is unchanged under rotation, but the footprint in graph 4 is of length 5 (love that 3-4-5 triangle!). Thus 2*a* – 5 and *a* = 5/2.

Finding *b*

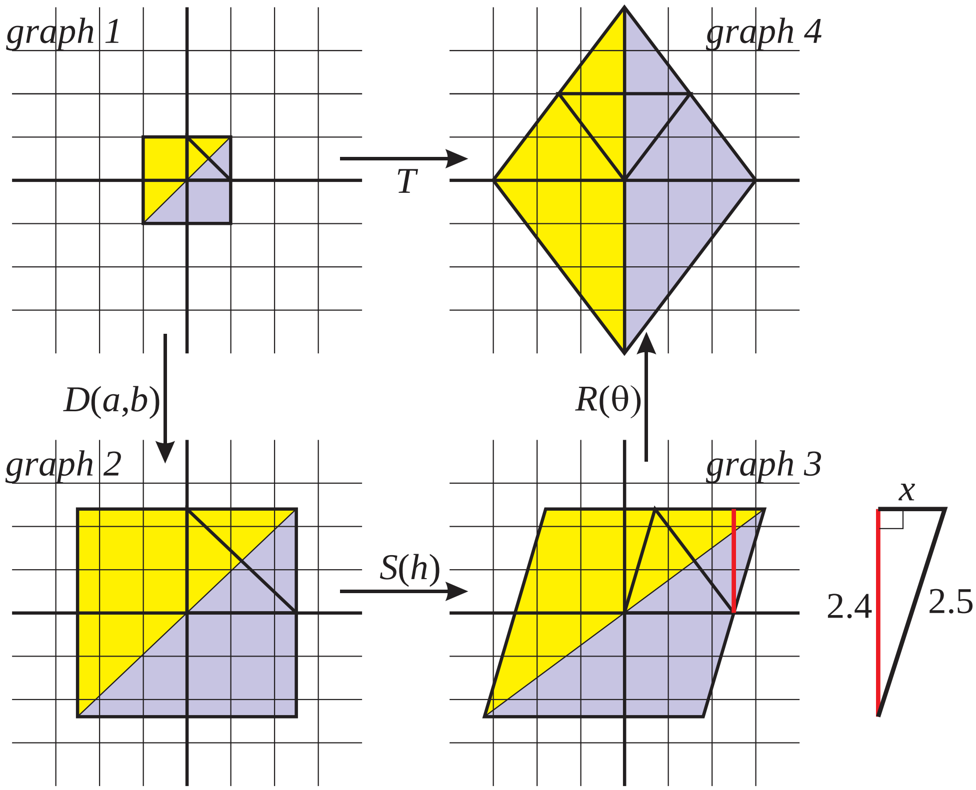


The vertical red line in graph 1 is height 1, half the height of the box. Its footprint in graph 2 is then the *y*-multiplier *b*. Since *S* preserves vertical distances, this is also the height of the top half of the graph-3 box. The footprint of this line in graph 4 can then be measured. It is one of the legs of a triangle whose opposite angle is θ with hypotenuse 3. Thus $\frac{b}{3} = \sin\theta$ and

$$b = 3\sin\theta = 3\frac{4}{5} = \frac{12}{5} = 2.4$$



Finding *h*



Finally we calculate the shear parameter *h*. In graph 3 I have drawn a vertical red line whose height is *b* which we have just calculated to be 2.4. The line is one leg of a right-angled triangle whose hypotenuse is half the side-length of the parallelogram (which is actually a rhombus as all sides are equal). These side-lengths are all 5 (seen in graph 4) so the half-length is 2.5. The top of the triangle is the distance *x* that the top of the box has moved under the shear *S*(*h*). Now recall that *h* is the sideways movement of a line of height $y = 1$. The top of the box is at height at $y = b$ so it will move a distance *hb*. Thus $x = hb$ and solving this for *h*:

$$h = \frac{x}{b}.$$

Now we have calculated $b = 2.4$ and we can use Pythagoras to find *x*:

$$2.5^2 = x^2 + 2.4^2$$

$$x^2 = 2.5^2 - 2.4^2 = (2.5 + 2.4)(2.5 - 2.4) = (4.9)(0.1) = 0.49$$

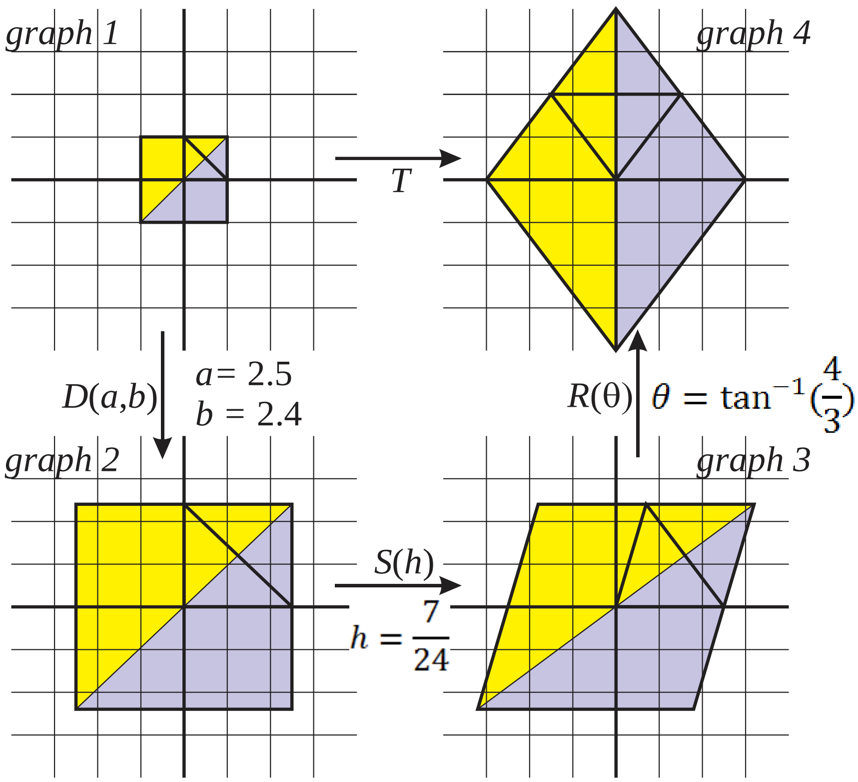
$$x = \sqrt{0.49} = 0.7$$

Finally:

$$h = \frac{x}{b} = \frac{0.7}{2.4} = \frac{7}{24}.$$

I normally use fractions rather than decimals unlike all my students who pull out their calculator (or frequently their cell phone) for the slightest reason. For this calculation, decimals work well especially if we use the nice difference of squares formula. We are of course lucky that it gives us an exact answer for *x*.

Putting it all together:



Example 6 algebra

Using algebraic reasoning, find the parameters a, b, h and θ

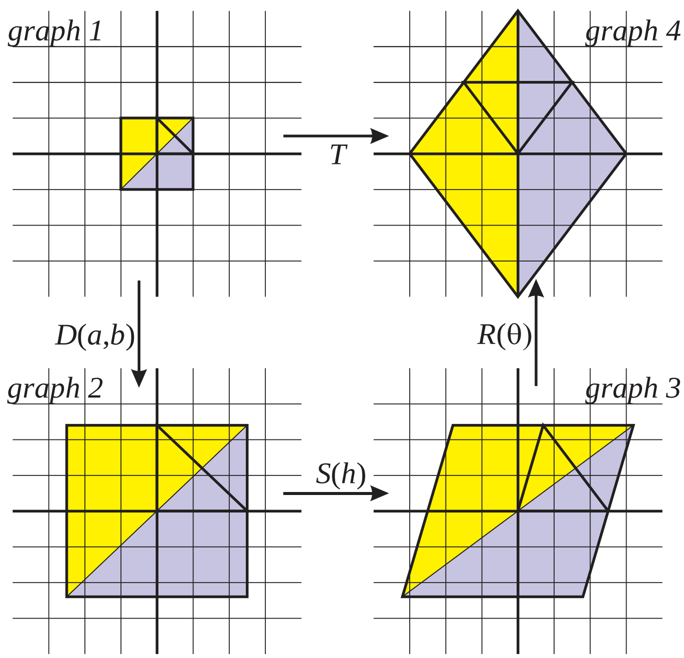
$$T = [R(\theta)] \cdot [S(h)] \cdot [D(a,b)]$$

$$\begin{bmatrix} 1.5 & -1.5 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$\begin{bmatrix} 1.5 & -1.5 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} ac & chb - sb \\ as & shb + cb \end{bmatrix}$$

Where $s = \sin\theta$ and $c = \cos\theta$, giving us 4 equations in 4 unknowns:

(1) $ac = 1.5$
(2) $as = 2$
(3) $chb - sb = -1.5$
(4) $shb + cb = 2$



Divide (2) by (1): $\frac{as}{ac} = \frac{s}{c} = \frac{2}{1.5} = \frac{4}{3}$

Hence $\tan\theta = \frac{4}{3}$ and therefore

$s = \frac{4}{5}$ and $c = \frac{3}{5}$

Now factor b out of (3) and (4):

$b(ch - s) = -1.5 \qquad b = \frac{-1.5}{ch-s}$

$b(sh + c) = 2 \qquad b = \frac{2}{sh+c}$

Setting these equal:

$$\frac{-1.5}{ch - s} = \frac{2}{sh + c}$$

Multiply by 2 and cross-multiply:

$$-3(sh + c) = 4(ch - s)$$

$$4s - 3c = 4ch + 3sh$$

$$h(4c + 3s) = 4s - 3c$$

$$h = \frac{4s - 3c}{4c + 3s} = \frac{4(4) - 3(3)}{4(3) + 3(4)} = \frac{7}{24}$$

Finally we get b:

$$b = \frac{2}{sh + c} = \frac{2 \cdot 5}{4\left(\frac{7}{24}\right) + 3} = \frac{10}{\left(\frac{7}{6}\right) + 3} = \frac{60}{7 + 6 \cdot 3} = \frac{60}{25} = \frac{12}{5}$$

I found that most of the grade 10 students were quite poor at what I call technical fluency, particularly in the manipulation of algebraic equations. Thus one of my objectives is to work on this with them and the example at the right illustrates a number of significant skills. For example note my decision to work with variables *s* and *c* until the end (to keep things simple and clear) and then use the homogeneity of the expression to "see" that the 5's in the denominators will all cancel and can be ignored.

Example 6 worksheet