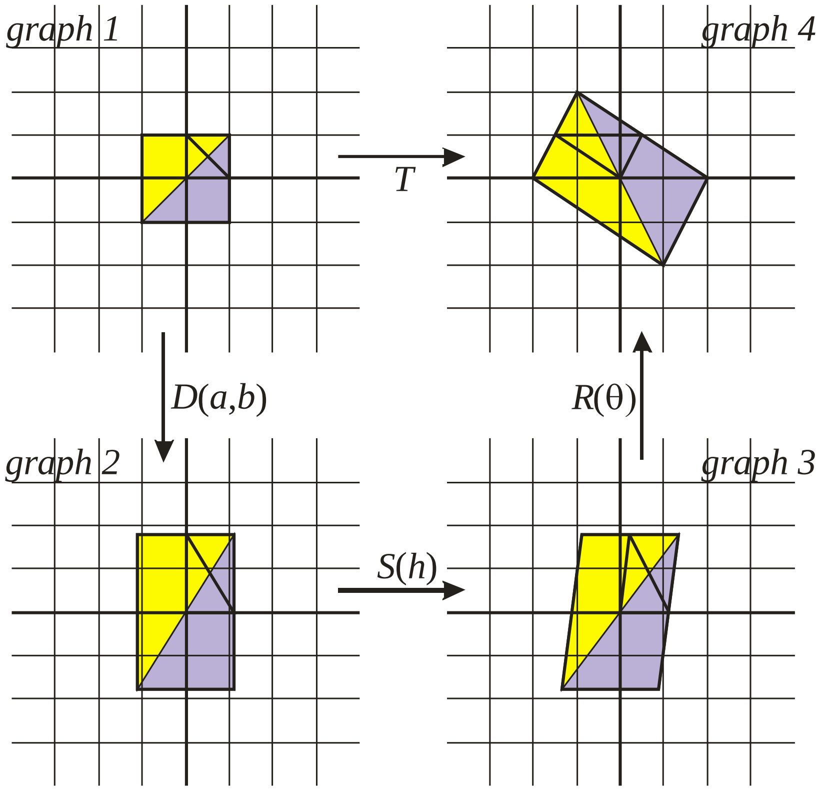
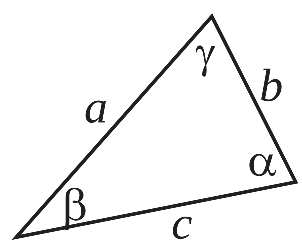
Example 7 Geometry

This has the same structure—a dilation, a shear, and then a rotation. Find the four parameters.



The one is quite a journey and is reserved for the ambitious students. What particularly interested me about this example is that something a bit extra seems to be needed to find the *y*-dilation *b*, and in fact there are different ways to get the job done. The two most accessible ways seem to be the *cosine law* which generalizes the Pythagorean Theorem and uses the side-lengths and one angle

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$



and the *sine law* which says that the ratio of the sine of each angle to the length of the opposite side is the same for all three angles:

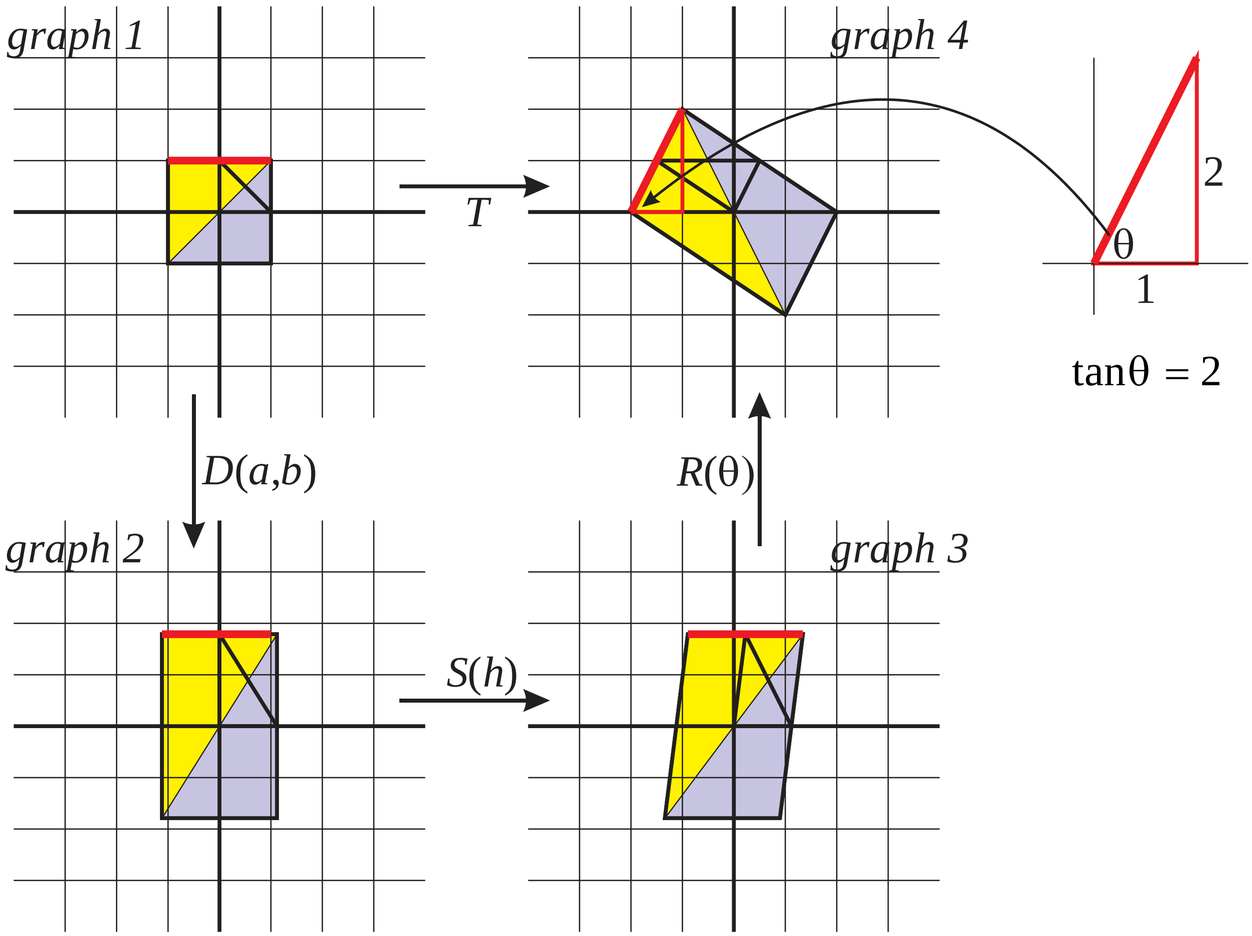
$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

and as a bonus both of these are on the grade 10 curriculum. I have rarely had occasion to use the sine law in my work so it was delightful to see it arise here.

Both of those solutions are given below.

Finding θ.

Our standard invariance ideas will give us the angle of rotation.



Start at graph 1. The top of the box is horizontal (using the grid points). Since horizontal lines remain horizontal under dilation, the top of the box in graph 2 is still horizontal. Since the shear simply moves horizontal lines sideways, the top of the graph 3 box is still horizontal. Thus the angle θ of rotation is the angle its footprint in graph 4 makes with the horizontal.

This angle is marked in the graph 4 blow up. Clearly

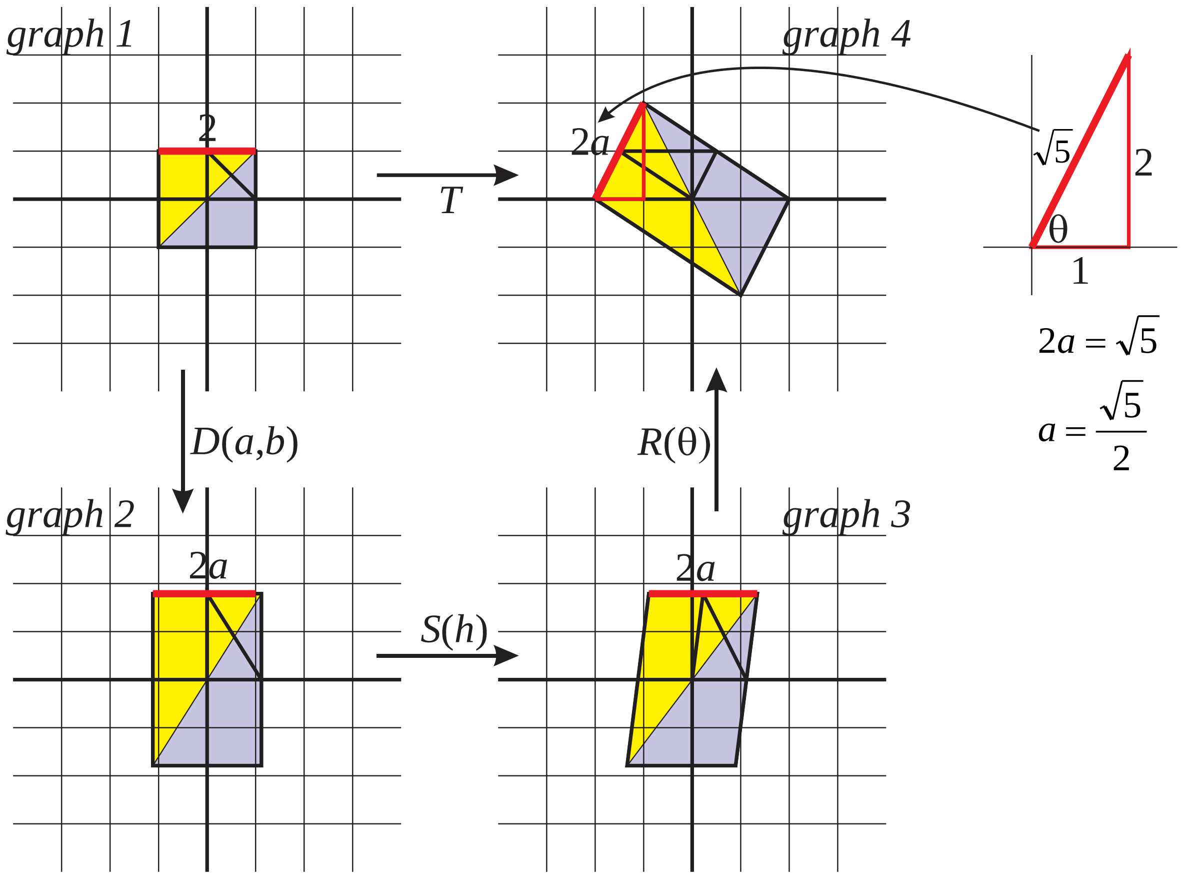
$$\tan\theta = 2$$

$$\theta = \arctan(2) \sim 63.4^{o}$$

I use the positive sign as the rotation is counterclockwise. From the diagram, that seems to be about right.

Finding a.

While we have the horizontal top of the box in our thoughts we will turn to the *a*-analysis just as we did in Example 1.



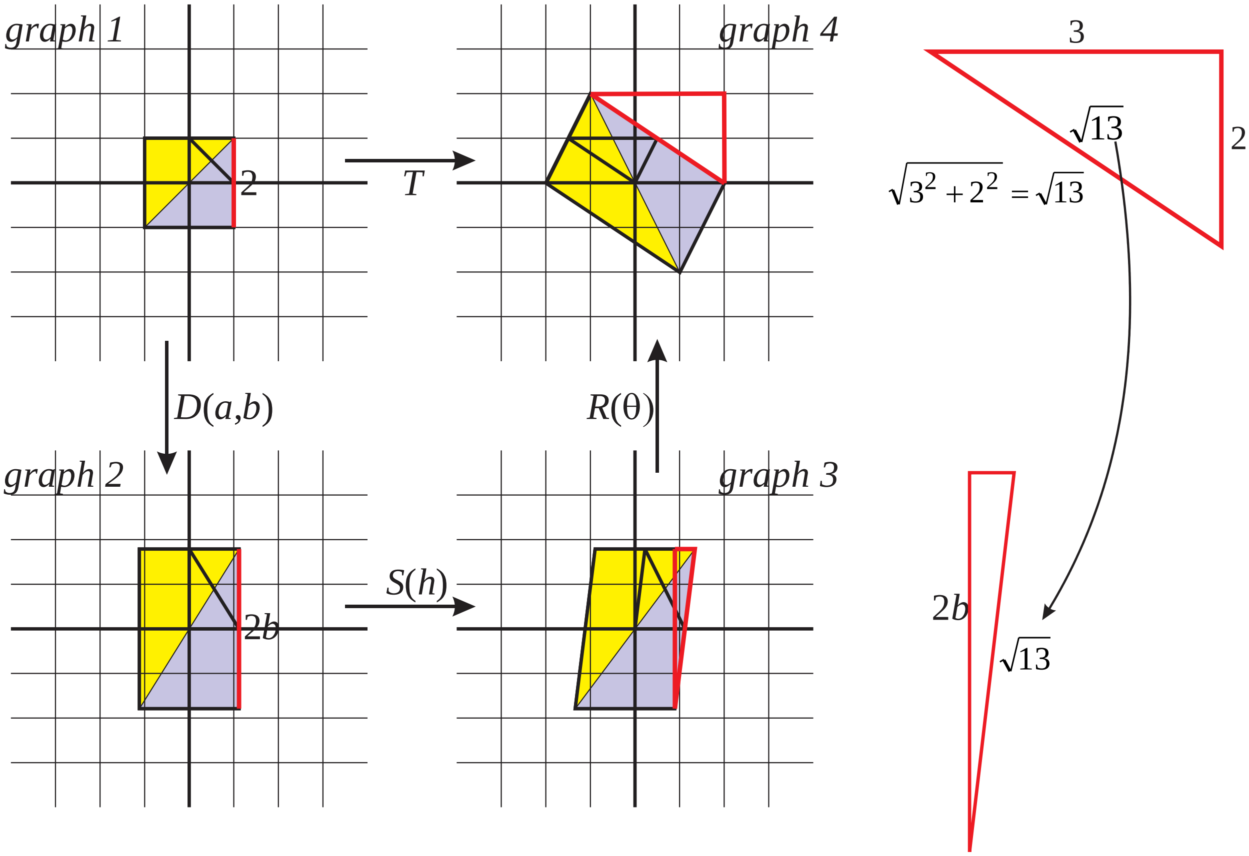
In graph 1 the top of the box is horizontal and has length 2. After the dilation, the top of the box in graph 2 will be horizontal with length 2*a*. In graph 3 it still has length 2*a* because the shear does not change the length of horizontal lines. Finally its footprint in graph 4 is also of length 2*a* because rotations do not change the length of any line.

We can measure the length of the footprint in graph 4—it's the hypotenuse of a right-angled triangle with legs of length 1 and 2 so has length $\sqrt{5}$. This is 2*a*. Thus

$$a = \frac{\sqrt{5}}{2}$$

Finding b.

This time we need to keep track of the length of a *vertical* line as it jumps from graph 1 to graph 2. I choose the right-hand side of the box.

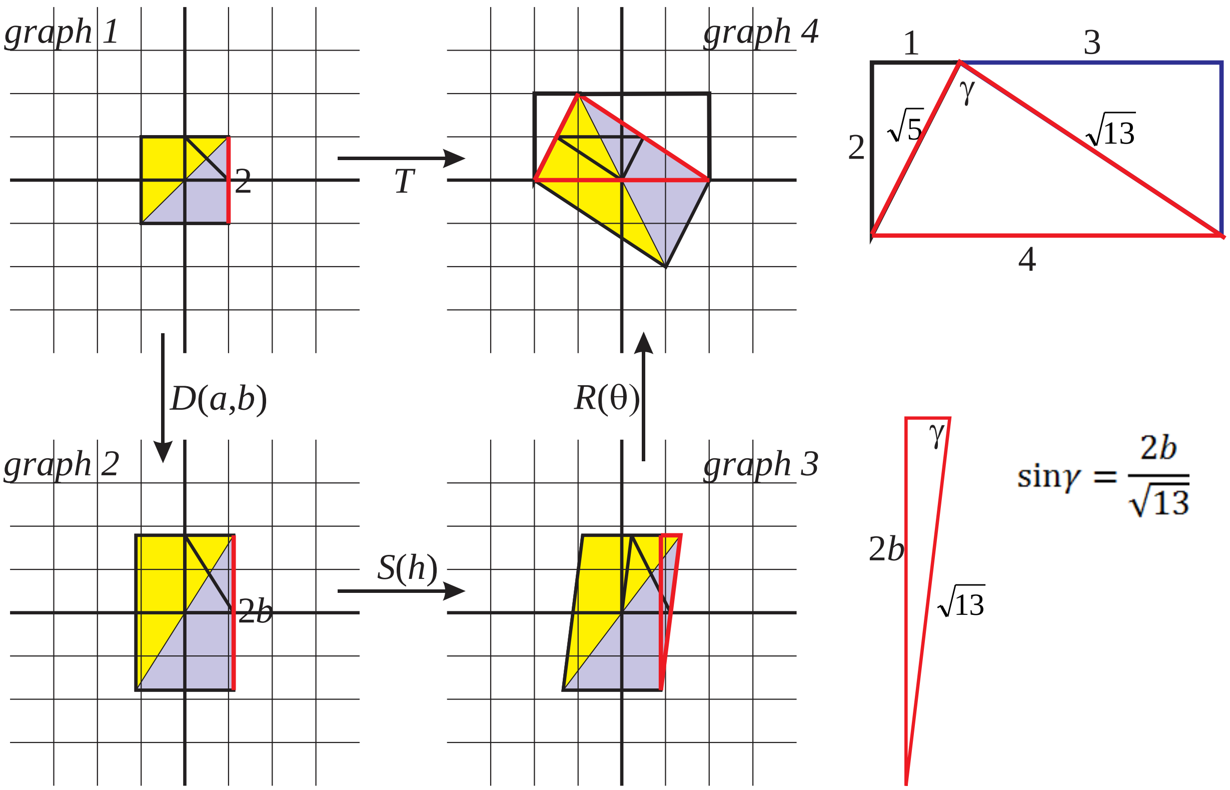


In graph 1 it has length 2; in graph 2 it has length 2*b*. Now in graph 3 we have to be careful—the shear actually increases the length of the side of the box. However the vertical distance between horizontal lines does not change as the shear only moves horizontal lines sideways. Thus any vertical line from the top down to the bottom is of length 2*b*. The red triangle in the graph 3 blow-up has such a vertical line as one of its sides and the hypotenuse of that triangle is the right side of the box. Can we find the length of that hypotenuse? Yes we can—its footprint in graph 4 has the same length (rotation) and is seen to be the hypotenuse of a right triangle with legs 2 and 3, thus it has length $\sqrt{13}$.

Okay. Where does the argument go from here? How are we to find *b*? We need to have one more piece of information about that skinny triangle in graph 3.

Finding b (cont'd)

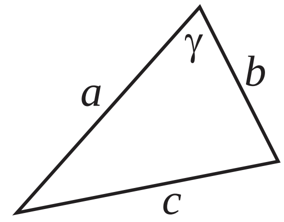
There is more than one way to proceed from here, but the way I have chosen is to look for the angle marked γ in graphs 3 and 4. In graph 3 we have $\sin\gamma = \frac{2b}{\sqrt{13}}$ so if we knew sinγ we could solve that equation for *b*.

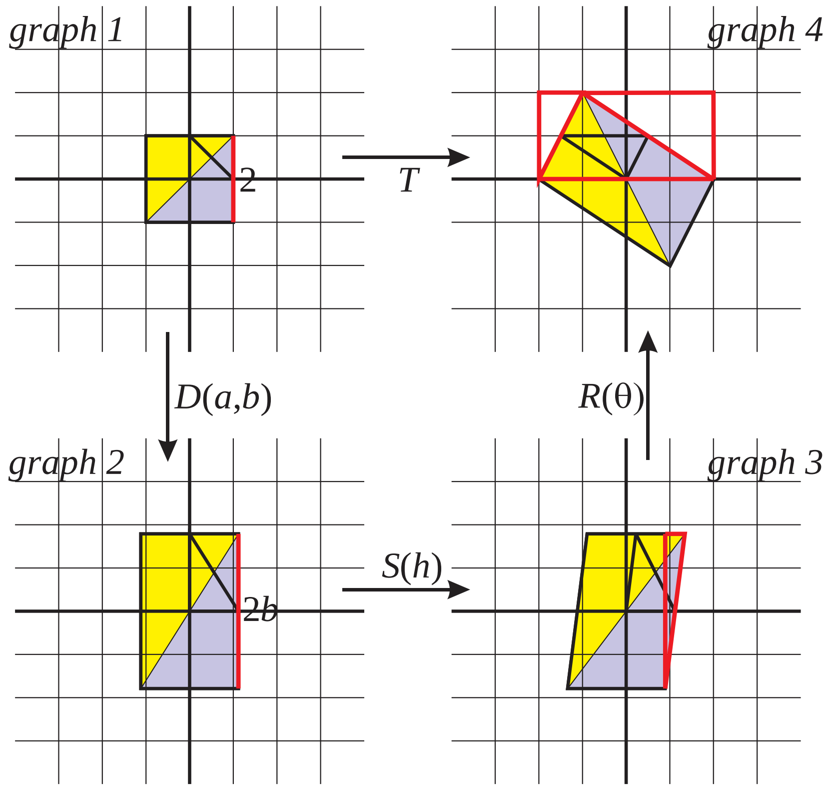


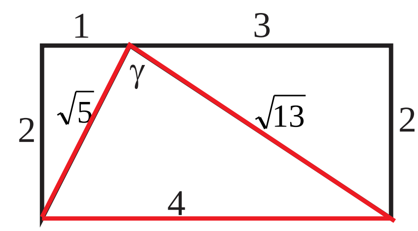
But if we look at the corresponding angle γ in graph 4, we see that it is one of the angles of a triangle all of whose sides we know or can work out (using Pythagoras). Can we find γ from that?

Yes we can and there are two ways we might proceed. One is to use the cosine law and the other is to use the sine law.

Finding b (using the cosine law) $c^2 = a^2 + b^2 - 2ab\cos\gamma$





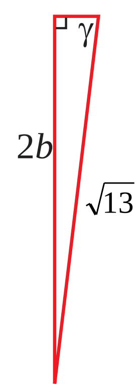


$$4^2 = \sqrt{5}^2 + \sqrt{13}^2 - 2\sqrt{5}\sqrt{13}\cos\gamma$$

$$16 = 5 + 13 - 2\sqrt{65}\cos\gamma$$

$$2\sqrt{65}\cos\gamma = 18 - 16 = 2$$

$$\cos\gamma = \frac{1}{\sqrt{65}}$$



$$\sin\gamma = \frac{2b}{\sqrt{13}}$$

Using the cosine law on the triangle in graph 4 gives us $\cos\gamma = \frac{1}{\sqrt{65}}$. Given that, what is sinγ? A simple answer is to draw a triangle with an angle γ whose cos is $\frac{1}{\sqrt{65}}$ (see diagram at the right), and then use Pythagoras to get the remaining side k. We get $k^2 + 1^2 = 65$ and that solves to give $k = 8$. That gives us

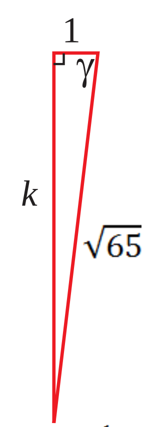
$$\sin\gamma = \frac{8}{\sqrt{65}}$$

But now going back to graph 3,

$$\sin\gamma = \frac{2b}{\sqrt{13}} = \frac{8}{\sqrt{65}}$$

and this solves to give

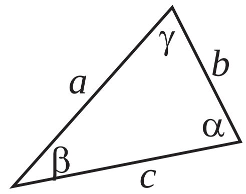
$$b = \frac{4}{\sqrt{5}}$$



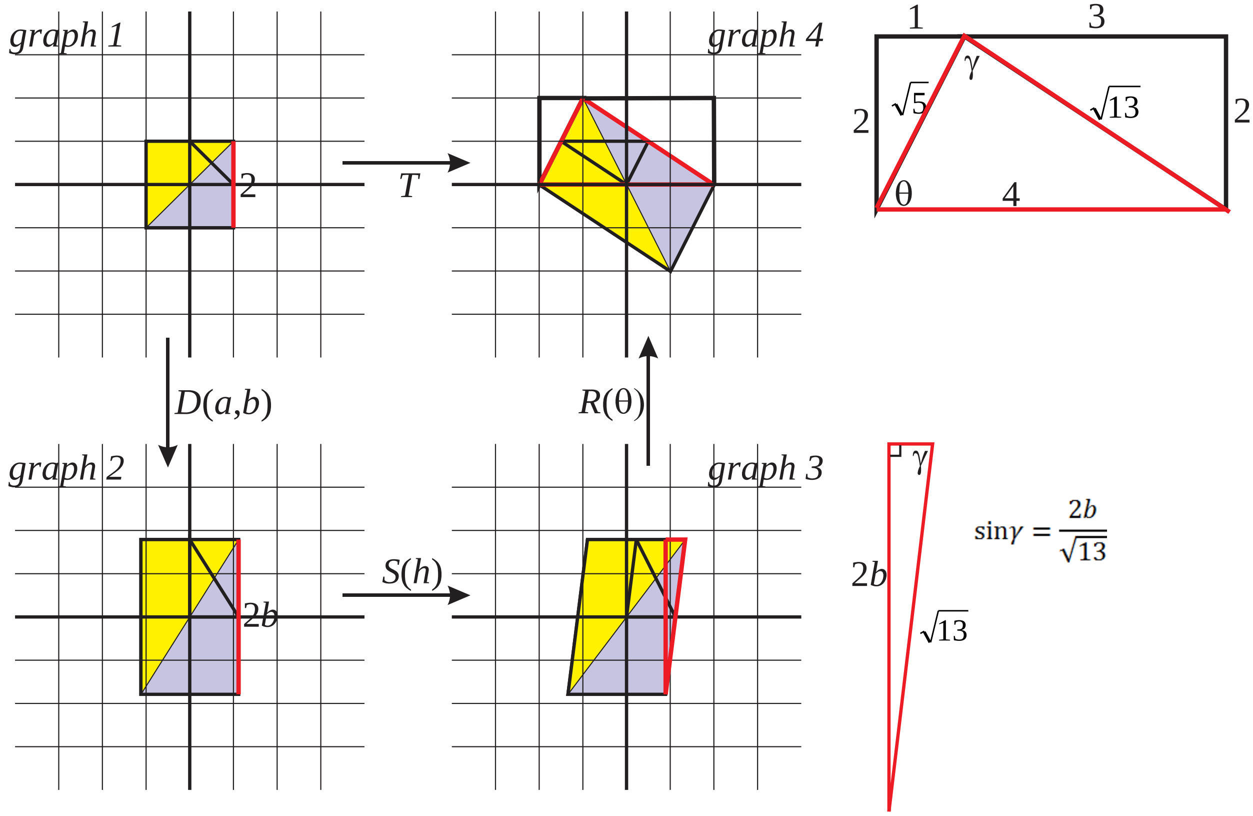
$$\cos\gamma = \frac{1}{\sqrt{65}}$$

Finding b (using the sine law)

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}$$



The sine law is a bit more direct but it requires knowledge of one of the other angles of the triangle in graph 4. Well the left-hand base angle is the rotation angle θ.

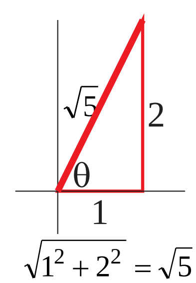


Using these two angles, the sine law gives us

$$\frac{\sin\gamma}{4}=\frac{\sin\theta}{\sqrt{13}}$$

Now we know that $\tan\theta = 2$, and hence (from the triangle at the right)

$$\sin\theta=\frac{2}{\sqrt{5}}$$



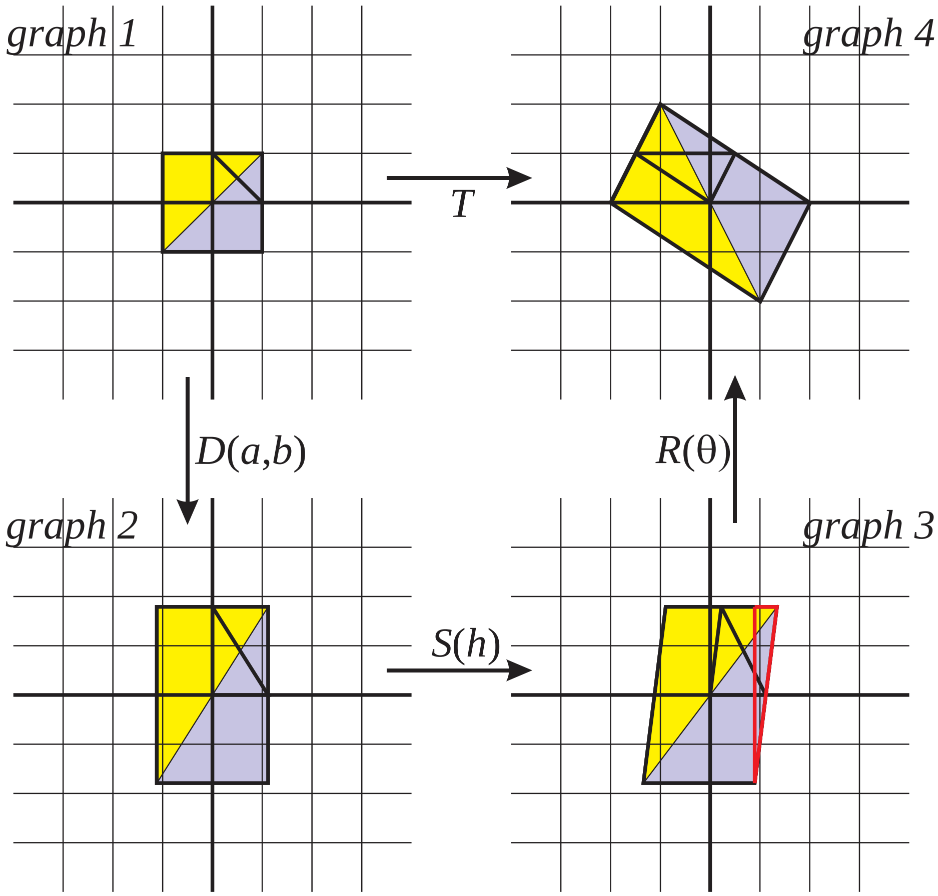
Putting these together:

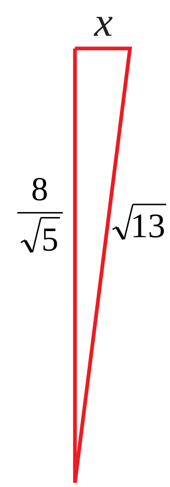
$$\sin\gamma = 4\frac{\sin\theta}{\sqrt{13}}=\frac{4\cdot 2}{\sqrt{5}\sqrt{13}}=\frac{8}{\sqrt{65}}$$

and then as above, we get $b=\frac{4}{\sqrt{5}}$.

Finding h.

Finally we look for the shear parameter h and I will use the angle approach. It's immediately clear that the shear angle σ is the bottom angle of that skinny red triangle.





We already have two sides of the skinny triangle, the hypotenuse $\sqrt{13}$ and the vertical side $2b = 2\dfrac{4}{\sqrt{5}} = \dfrac{8}{\sqrt{5}}$, so we can find the remaining side x by Pythagoras,

$$x^2 + \left(\frac{8}{\sqrt{5}}\right)^2 = \sqrt{13}^2$$

We solve this (box at right) to get $x = \dfrac{1}{\sqrt{5}}$. This is twice the movement of the top of the box. Now the top of the box is at height b so it moves a distance bh. Thus

$$x = 2bh = \frac{1}{\sqrt{5}}$$

$$h = \frac{1}{2\sqrt{5}b} = \frac{1}{2\sqrt{5}}\frac{\sqrt{5}}{4} = \frac{1}{8}$$

$$x^2 + \left(\frac{8}{\sqrt{5}}\right)^2 = \sqrt{13}^2$$

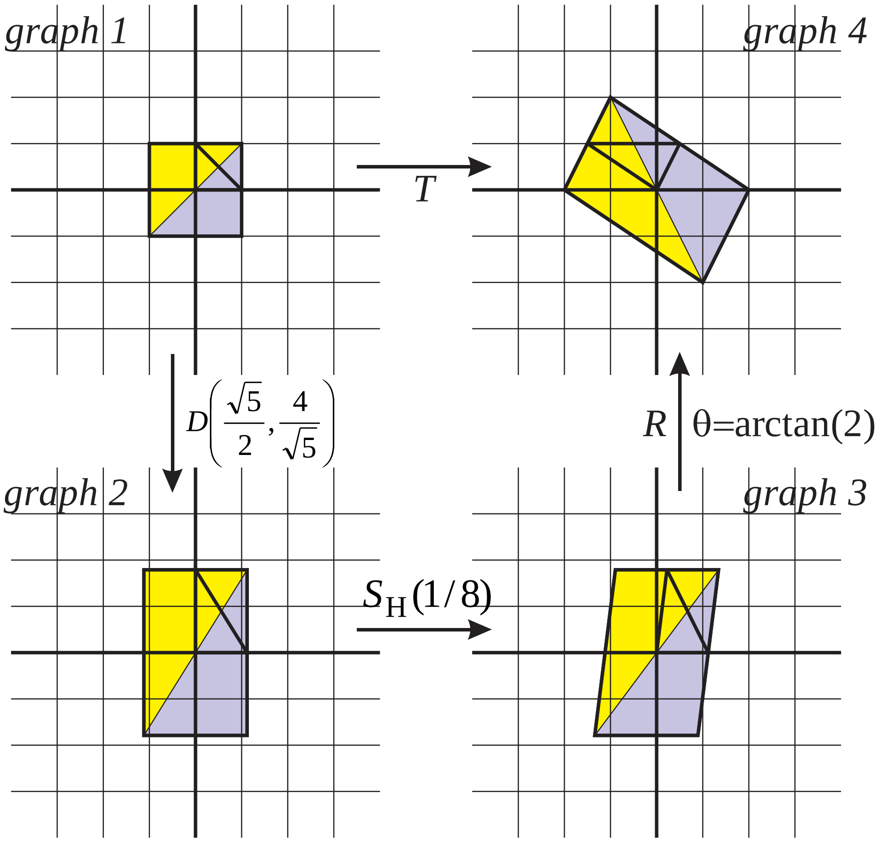
$$x^2 + \frac{64}{5} = 13$$

$$x^2 = 13 - \frac{64}{5}$$

$$= \frac{65}{5} - \frac{64}{5} = \frac{1}{5}$$

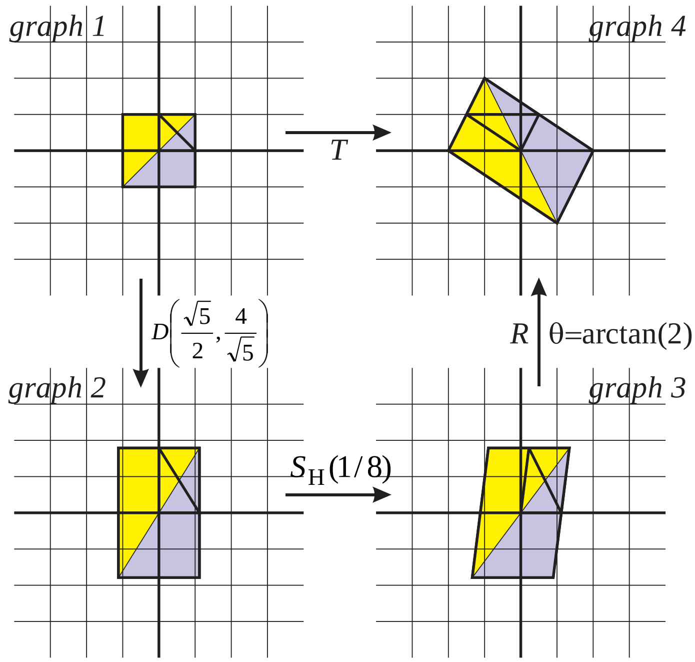
$$x = \frac{1}{\sqrt{5}}$$

The final answer is presented below.



That was quite a journey.

Example 7 Algebra



Find θ, a, b and h such that

$$T = R(\theta)\circ S(h)\circ D(a,b)$$

$$\begin{bmatrix}1/2 & -3/2\\ 1 & 1\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\cdot\begin{bmatrix}1 & h\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}$$

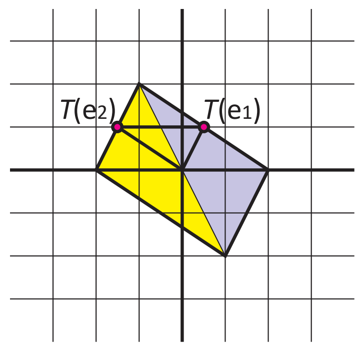
To simplify notation, set $s = \sin\theta$ and $c = \cos\theta$

$$\begin{bmatrix}1/2 & -3/2\\ 1 & 1\end{bmatrix}=\begin{bmatrix}c & -s\\ s & c\end{bmatrix}\cdot\begin{bmatrix}a & hb\\ 0 & b\end{bmatrix}$$

$$\begin{bmatrix}1/2 & -3/2\\ 1 & 1\end{bmatrix}=\begin{bmatrix}ca & chb-sb\\ sa & shb+cb\end{bmatrix}$$

giving us 4 equations in 4 unknowns:

(1) $ca = 1/2$
(2) $sa = 1$
(3) $chb - sb = -3/2$
(4) $shb + cb = 1$

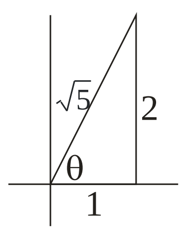


Divide (2) by (1) to get $\tan\theta = \dfrac{s}{c} = \dfrac{1}{1/2} = 2$

hence $\theta = \tan^{-1}(2)$

From the diagram, this is clearly a positive angle in quadrant 1:

$$s = \sin\theta = 2/\sqrt{5}$$
$$c = \cos\theta = 1/\sqrt{5}$$



From (1) we then get $a = \dfrac{1}{s} = \dfrac{\sqrt{5}}{2}$.

From (3) $b = \dfrac{-3}{2(ch-s)} = \dfrac{-3\sqrt{5}}{2(h-2)}$

From (4) $b = \dfrac{1}{sh+c} = \dfrac{\sqrt{5}}{2h+1}$

$$\Rightarrow \frac{-3\sqrt{5}}{2(h-2)} = \frac{\sqrt{5}}{2h+1} \Rightarrow -3(2h+1) = 2(h-2) \Rightarrow h = \frac{1}{8}$$

Putting this into the equation $b = \dfrac{\sqrt{5}}{2h+1}$ gives us $b = \dfrac{\sqrt{5}}{2/8+1} = \dfrac{\sqrt{5}}{5/4} = \dfrac{4}{\sqrt{5}}$.

Example 6 worksheet