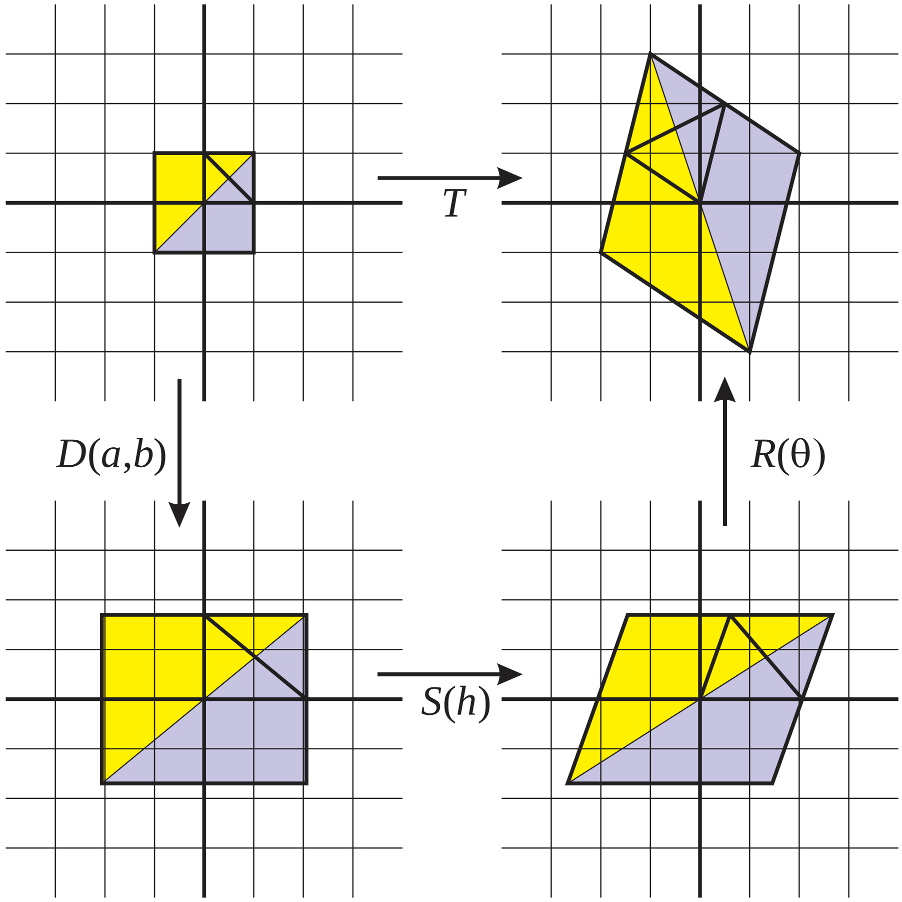
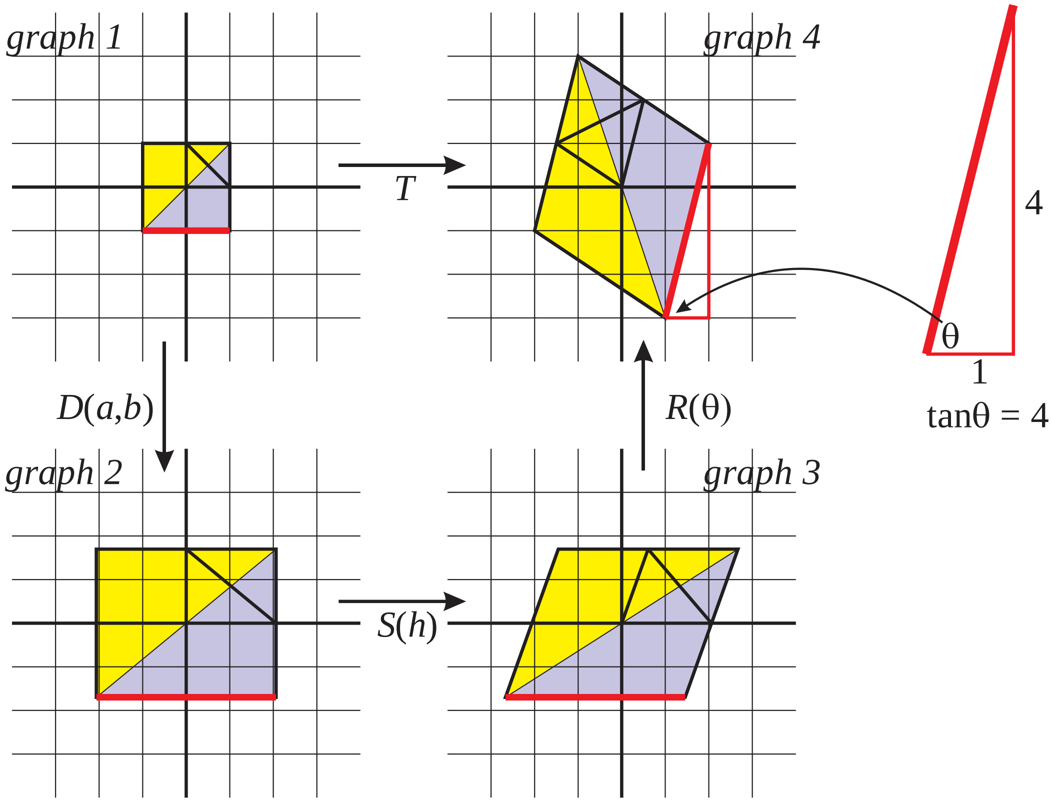
Example 8 geometry



Finding θ.

Again we can use our standard invariance ideas to get the angle.
This time I choose to use the bottom of the box.



The bottom of the box is horizontal in graph 1 and since horizontal lines remain horizontal under dilation, it is still horizontal in graph 2. Since the shear simply moves horizontal lines sideways, it is still horizontal in graph 3. Thus the angle θ of rotation is the angle its footprint in graph 4 makes with the horizontal.

This angle is marked in the graph 4 blow up. Clearly

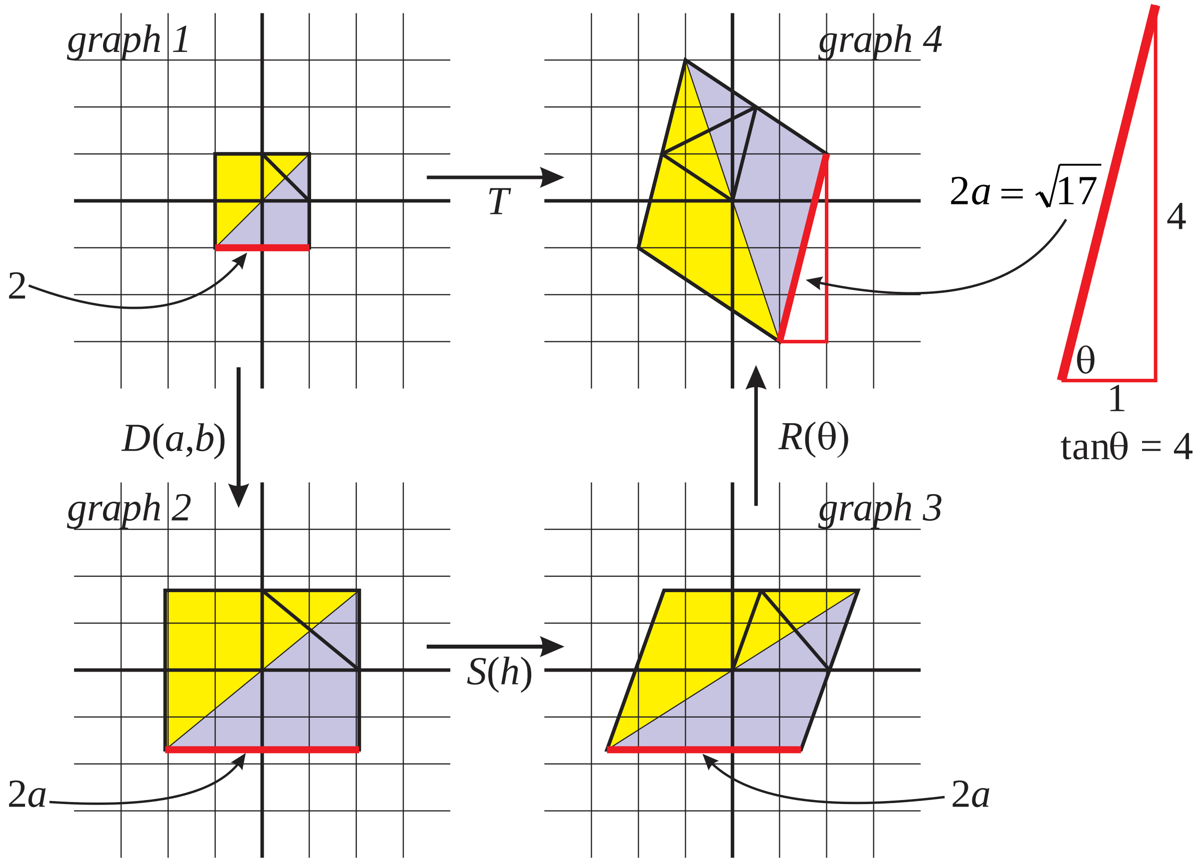
$$\tan\theta = 4.$$

$$\theta = \arctan(4) \sim 76.0^{\circ}$$

I use the positive sign as the rotation is counterclockwise. From the diagram, that seems to be about right.

Finding a.

While we have the horizontal bottom of the box in our thoughts we will turn to the *a*-analysis just as we did in Example 1.



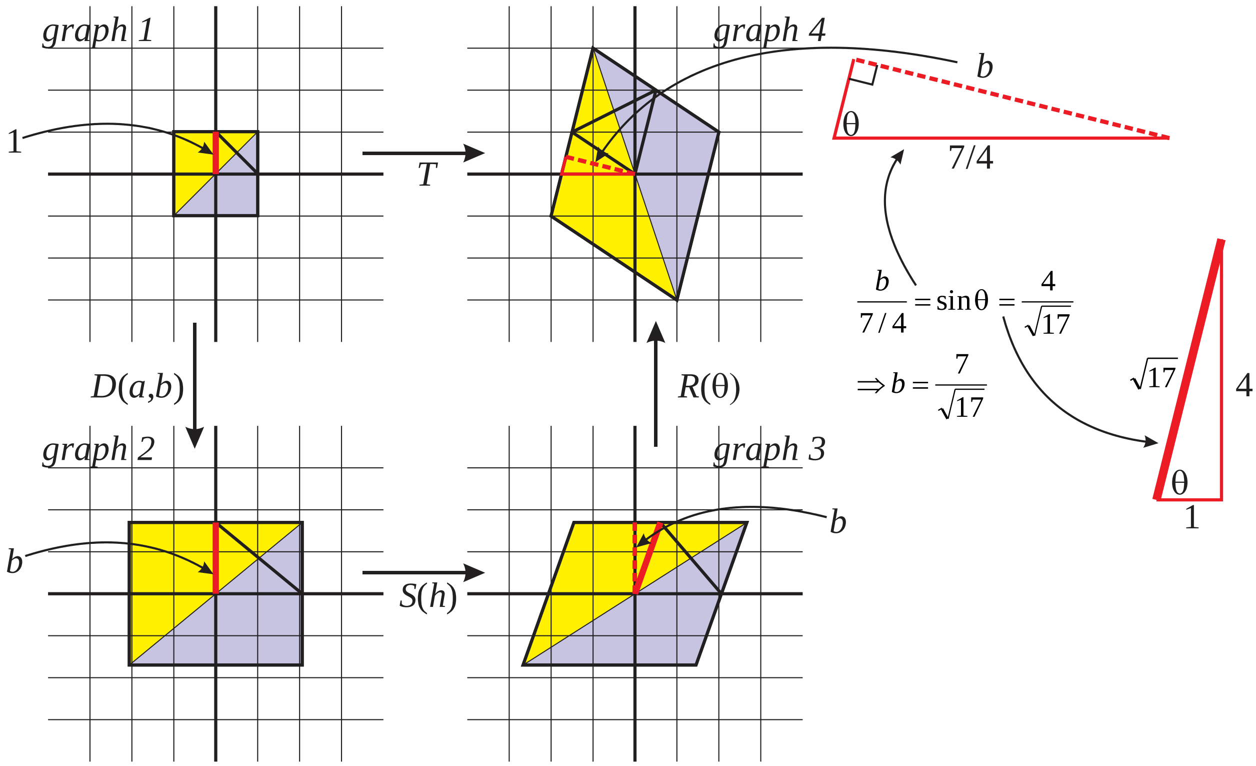
In graph 1 the bottom of the box is horizontal and has length 2. After the dilation, the bottom of the box in graph 2 will be horizontal with length 2*a*. In graph 3 it still has length 2*a* because the shear does not change the length of horizontal lines. Finally its footprint in graph 4 is also of length 2*a* because rotations do not change the length of any line.

We can measure the length of the footprint in graph 4—it's the hypotenuse of a right-angled triangle with legs of length 1 and 4 so has length $\sqrt{17}$. This is 2*a*. Thus

$$a = \frac{\sqrt{17}}{2}$$

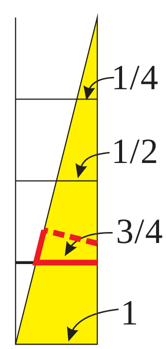
Finding b.

This time we need to keep track of the length of a *vertical* line as it jumps from graph 1 to graph 2. I choose the line from the origin to the top of the box.



In graph 1 the line has length 1; in graph 2 it has length *b*. Now in graph 3 its footprint is actually a bit longer, but since the vertical distance between horizontal lines does not change (as the shear only moves horizontal lines sideways), the vertical line from the origin to the top of the box (red *dotted*) still has length *b*. The footprint of that dotted line in graph 4 also has length *b* and since it is perpendicular to the left side of the box, it is one of the legs of a right triangle, whose opposite angle is the rotation angle θ. [Okay—do you see why that is: it's the angle between the horizontal and what in graph3 is the top of the box.] Now we know that $\tan\theta = 4$, so if we knew any side length of that small triangle, we'd be all set.

It took me a while to see this, but there's a simple argument that the hypotenuse of that small triangle is 7/4. The diagram at the right is a 4×1 rectangle extracted from graph4. Can you see how similar triangles can be used to deduce that the red line is of length 3/4?



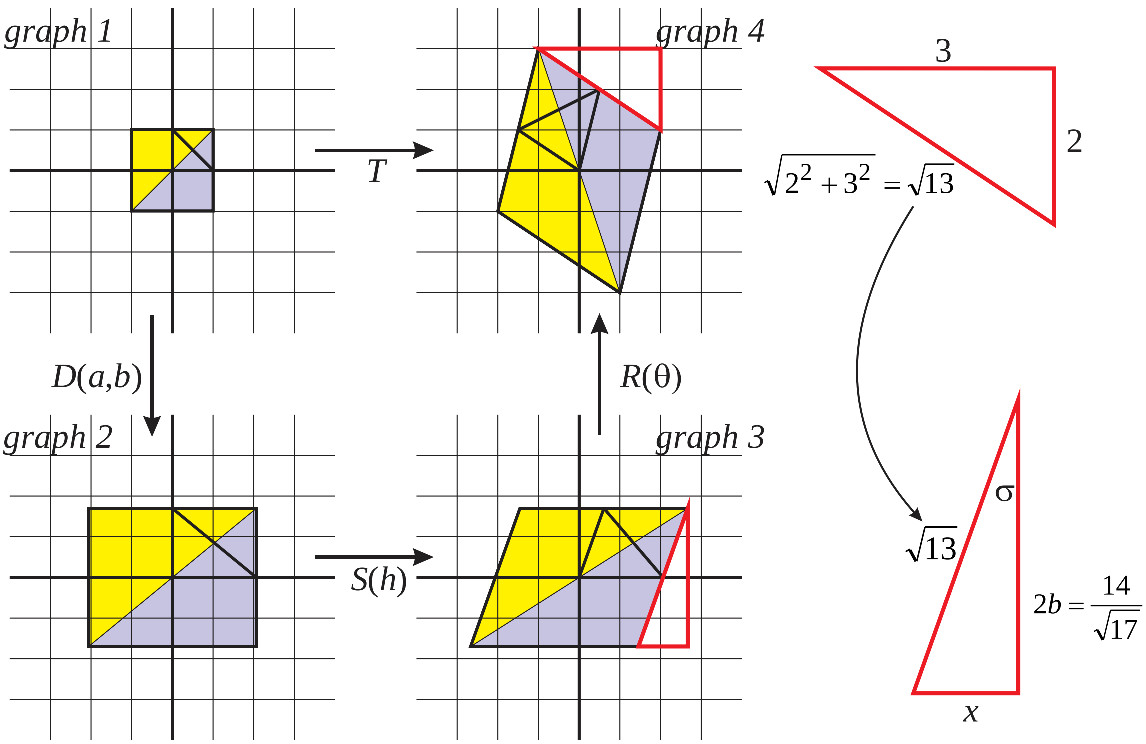
We conclude that

$$b = \frac{7}{\sqrt{17}}.$$

This simple approach was used in Example 6 and will also work in Example 7.

Finding h.

Finally, we look for the shear parameter h and I will use the angle approach. The shear angle σ is the top angle of the red triangle in graph3. If we knew two of the sides of the triangle, that would give us a way of finding σ.



Well first of all we know the vertical side: it is $2b=\frac{14}{\sqrt{17}}$.

Secondly the hypotenuse is found from its footprint on graph4 to be $\sqrt{13}$.
If we let x be the horizontal base of the triangle, then Pythagoras gives us:

$$x^2+\left(\frac{14}{\sqrt{17}}\right)^2=\sqrt{13}^2$$

We solve this (box at right) to get $x=\frac{5}{\sqrt{17}}$. Then

$$\tan\sigma=\frac{x}{14/\sqrt{17}}=\frac{5/\sqrt{17}}{14/\sqrt{17}}=\frac{5}{14}.$$

Finally we get $h=-\tan\sigma=-\frac{5}{14}$ but we see that h is in fact positive

(indeed σ is a negative angle) so that gives us $h=\frac{5}{14}$.

$$x^2+\left(\frac{14}{\sqrt{17}}\right)^2=\sqrt{13}^2$$

$$x^2+\frac{196}{17}=13$$

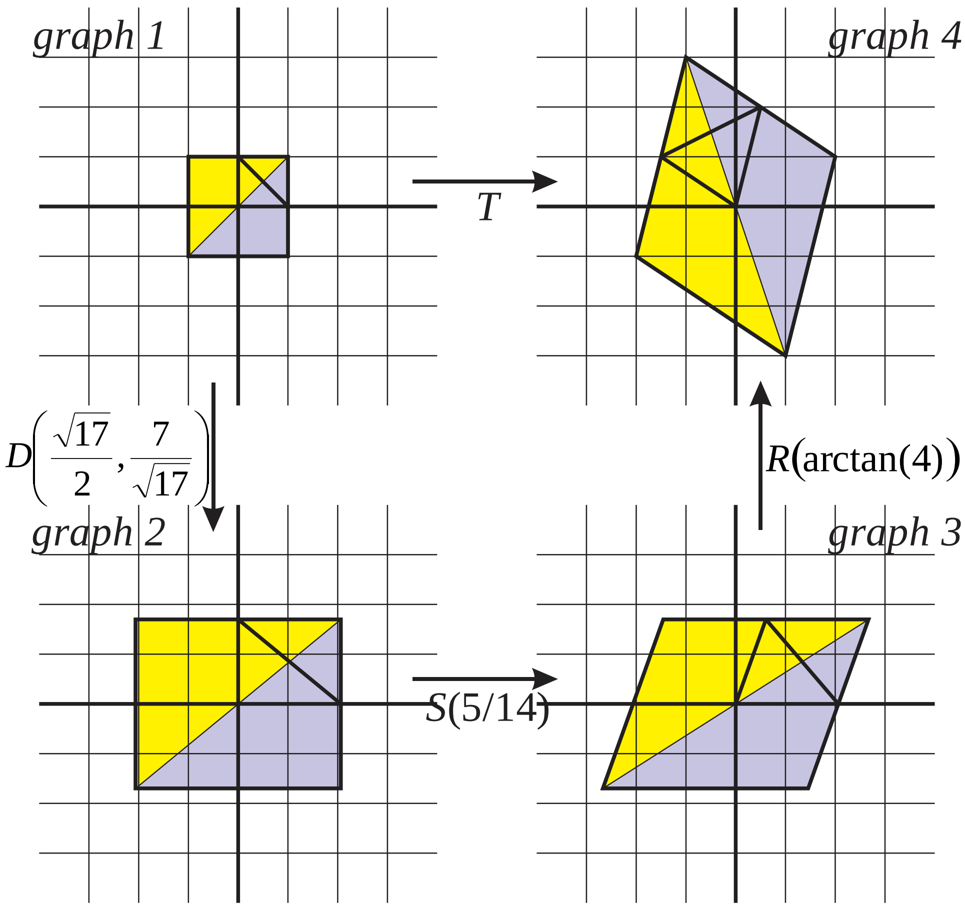
$$17x^2+196=221$$

$$17x^2=25$$

$$x^2=\frac{25}{17}$$

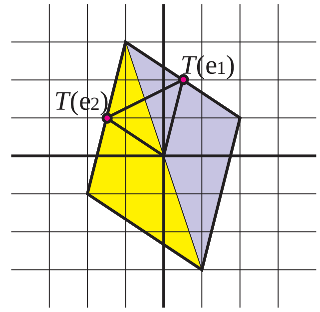
$$x=\frac{5}{\sqrt{17}}$$

The final solution is presented below.


graph 1
graph 4
T
$D\left(\frac{\sqrt{17}}{2}, \frac{7}{\sqrt{17}}\right)$
$R(\arctan(4))$
graph 2
graph 3
S(5/14)

Example 8 algebra

Find θ, a, b and h such that



$$T(\mathbf{e}_1) = (1/2, 2)$$
$$T(\mathbf{e}_2) = (-3/2, 1)$$

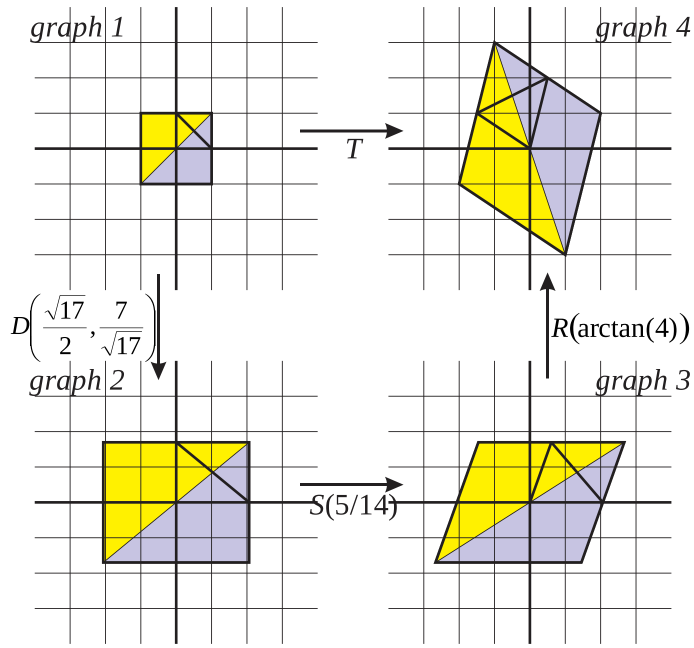
$$T = R(\theta) \circ S(h) \circ D(a,b)$$

$$\begin{bmatrix} 1/2 & -3/2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

To simplify notation, set $s = \sin\theta$ and $c = \cos\theta$

$$\begin{bmatrix} 1/2 & -3/2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \cdot \begin{bmatrix} a & hb \\ 0 & b \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & -3/2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} ca & chb - sb \\ sa & shb + cb \end{bmatrix}$$



giving us 4 equations in 4 unknowns:

(1) $ca = 1/2$

(2) $sa = 2$

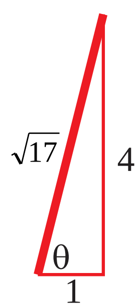
(3) $chb - sb = -3/2$

(4) $shb + cb = 1$

Divide (2) by (1) to get $\tan\theta = \frac{s}{c} = \frac{2}{1/2} = 4$

hence $\theta = \tan^{-1}(4)$

From the diagram, θ is clearly a positive angle in quadrant 1, hence:



$$s = \sin\theta = 4/\sqrt{17}$$
$$c = \cos\theta = 1/\sqrt{17}$$

From (2) we then get

$$a = \frac{2}{s} = \frac{\sqrt{17}}{2}.$$

From (3) $b = \frac{-3}{2(ch - s)} = \frac{-3\sqrt{17}}{2(h-4)}$

From (4) $b = \frac{1}{sh + c} = \frac{\sqrt{17}}{4h+1}$

$$\Rightarrow \frac{-3\sqrt{17}}{2(h-4)} = \frac{\sqrt{17}}{4h+1} \Rightarrow -3(4h+1) = 2(h-4) \Rightarrow h = \frac{5}{14}$$

Putting this into the bottom equation $b = \frac{\sqrt{17}}{4h+1}$ gives us $b = \frac{\sqrt{17}}{20/14+1} = \frac{\sqrt{17}}{34/14} = \frac{\sqrt{17}}{17/7} = \frac{7}{\sqrt{17}}$.

Example 6 worksheet