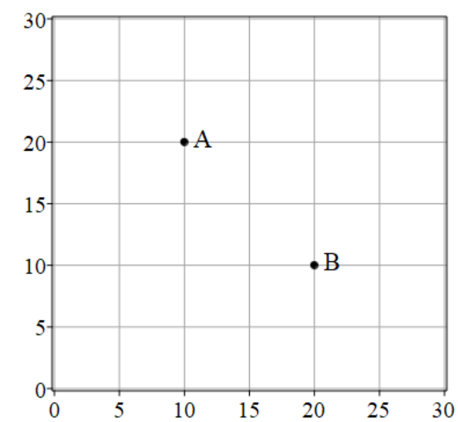
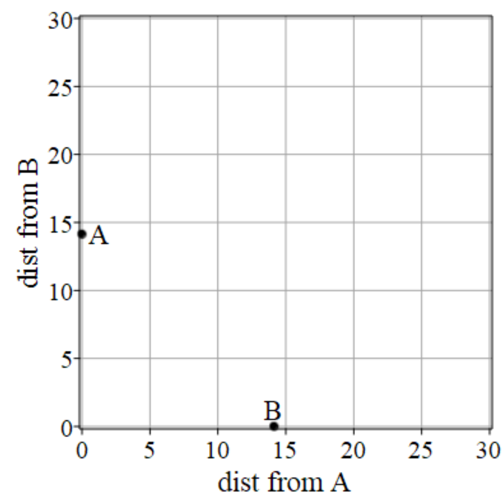
Two Towns

The landscape graph.
The graph at the right depicts a landscape—if you like think of the positive *x*-axis as East and the positive *y*-axis as North. There are two towns A and B at the coordinates (10, 20) and (20, 10).

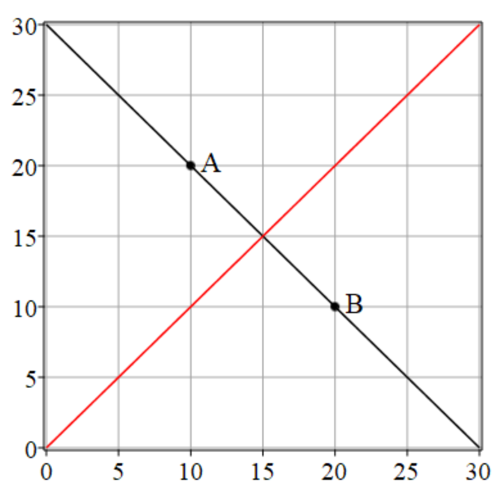


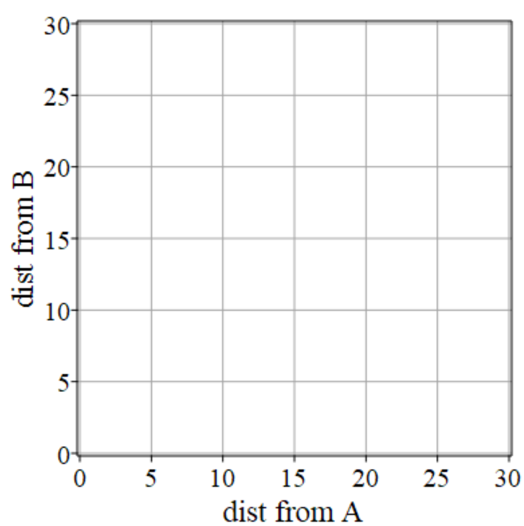
The distance graph.
It turns out that the significant information that we need to keep track of (for anyone in the landscape) is their distance from A and their distance from B. That information gets plotted on a new set of axes where distance from B is plotted against distance from A. What we need to be able to do is easily translate information from one graph to the other.



For example, the two towns have been plotted on this new graph. What exactly are their coordinates?
[These are $A(0, 10\sqrt{2})$ and $(10\sqrt{2}, 0)$]

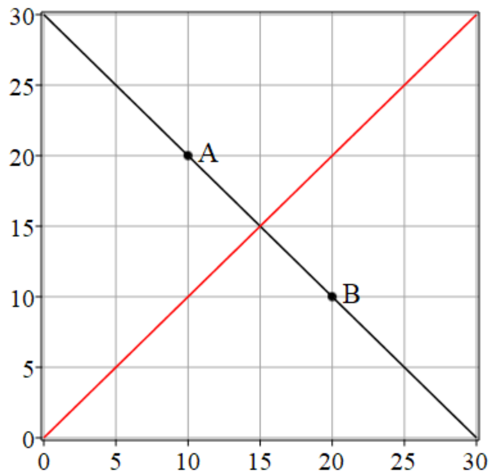
Example 1. A pair of lines are plotted on the landscape graph. Carefully plot them on the distance graph at the right.

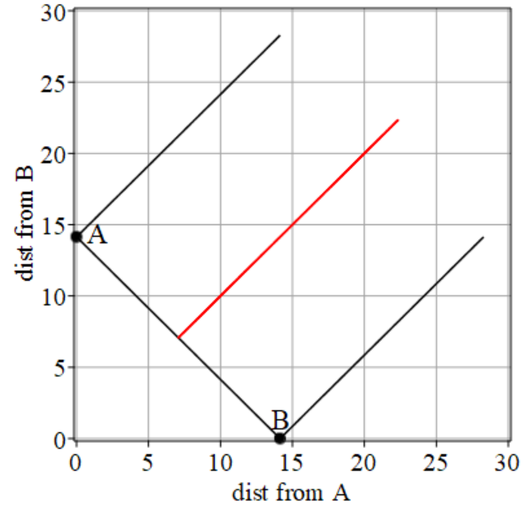




[The inspiration for this unit came from Paoletti and Moore (2018) A covariational understanding of function: putting a horse before the cart, *For the Learning of Mathematics* **38**, 3 pp37-43.

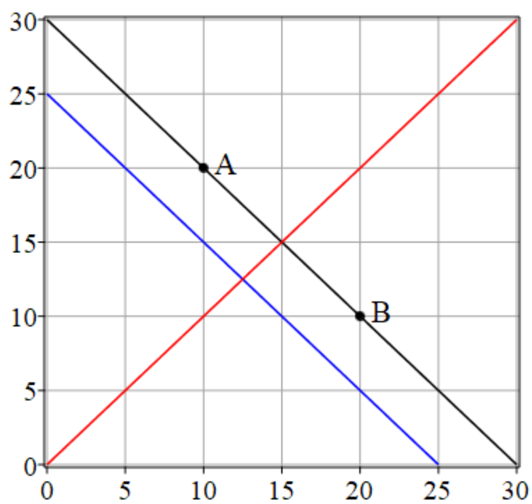
Solution to Example 1.

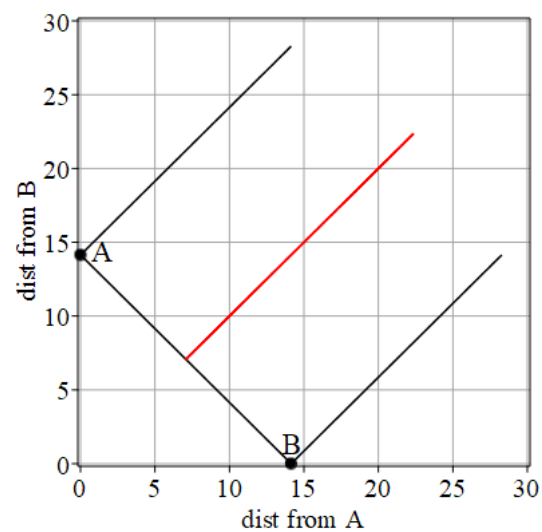




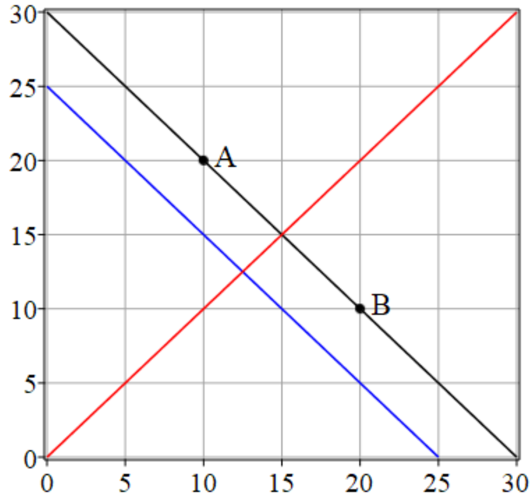
This first example is a bit of a “breakthrough” in terms of giving the students a feel for this new coordinate system. You can try to get the distance plot purely from graphical considerations, or you can try some algebra. For example consider the red line that passes through A and B. Let *x* and *y* be the distances of any point from A and B. Then for a point on the red line *between* A and B we have $x + y = \sqrt{200}$ and that’s a straight line on the distance graph of slope −1. But for a point on the red line on the other side of A we have $y = x + \sqrt{200}$ and that’s a line of slope +1. And we get a similar line segment on the other side of B. You need to think carefully but this is a wonderful graphical playground.

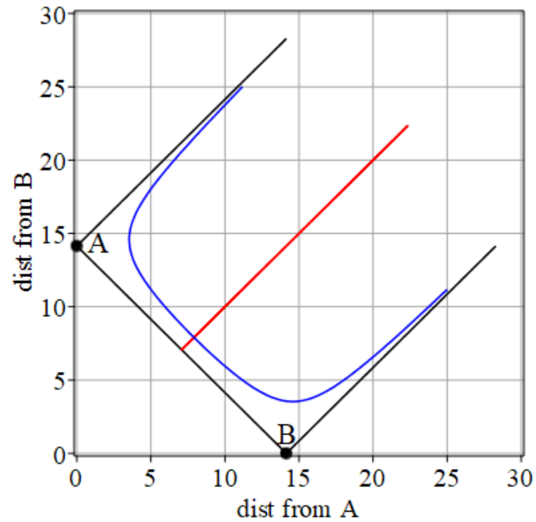
Example 2. Now how do we handle the blue line?





Solution to Example 2.

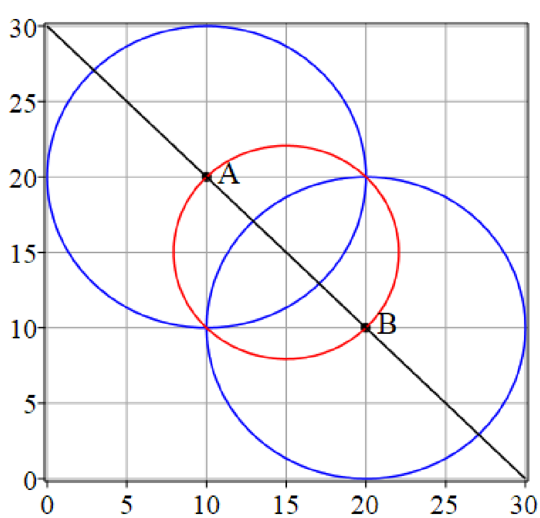


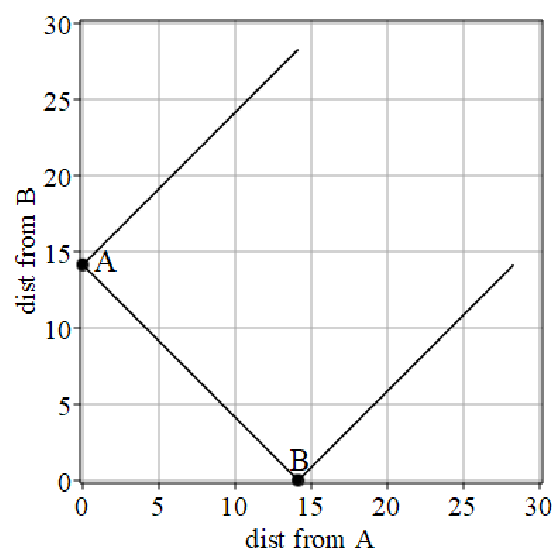


An interesting aspect of the new trajectory is that it is a smooth curve. Many students will draw it with two corners. It's hard to make a convincing argument that it can't have a corner without some calculus, but it's still worthwhile for students to see this and contemplate it. That's part of the process of developing their intuition.

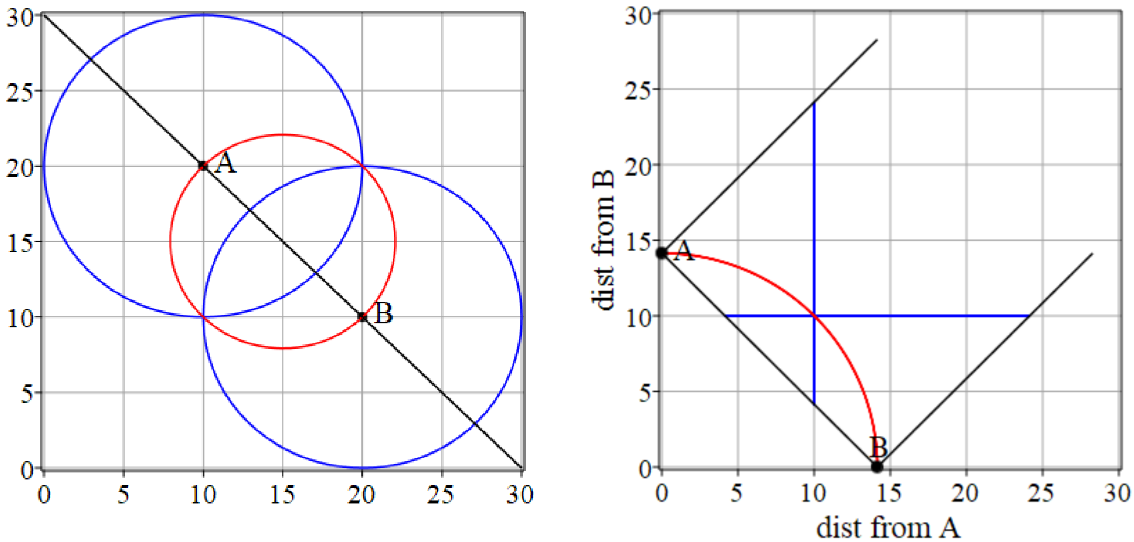
A nice exercise is to find the coordinates of the two "vertices" of the curve––the points that are closest to A and closest to B.
[The point closest to A is at $\left(\sqrt{50}/2, \sqrt{850}/2\right) \approx (3.5, 14.6)$.]

Example 3. Here are some circles on the landscape graph. Carefully plot them on the distance graph.



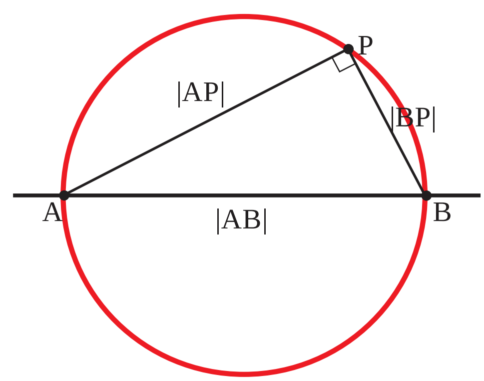


Solution to Example 3.



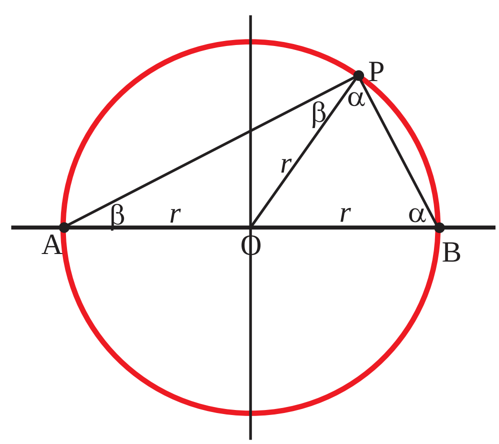
Goodness gracious! What on earth is that lovely red curve at the right? Could it possibly be a quarter circle? Or is that too much to ask. Offer a prize to the first student who settles that question.

Let's do some geometry. Here is a basic result in the geometry of circles. When I was a lad it was a standard (and wonderful) part of the Ontario Grade 10 curriculum, but I suspect it is now rarely found.



Proposition: The diameter of a circle subtends a 90° angle at any point on the circumference. In the diagram at the right, the angle BPA is 90° for any point P on the circle.

Proof.
Let O be the centre of the circle and take its radius to be r. The triangles BOP and AOP are isosceles so their base angles are equal. Take these to be β and α. The sum of the interior angles of the triangle APB is 180°:



$$2\alpha + 2\beta = 180.$$

It follows that

$$\alpha + \beta = 90$$

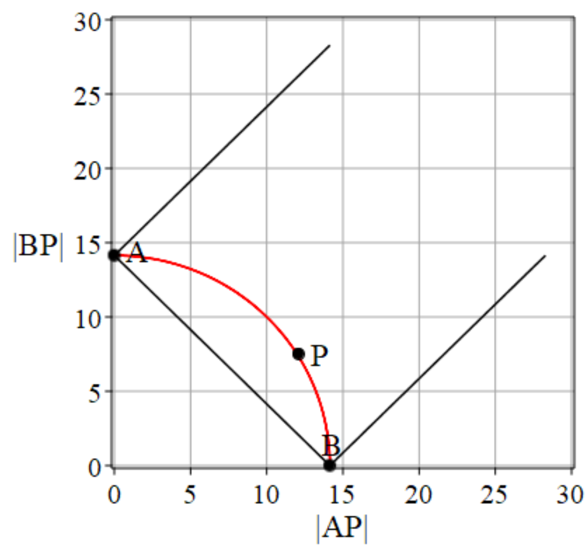
and that's our result.

Now what does this tell us about that red curve on the distance graph?

Look at the landscape graph. Since BPA is a right-angled triangle, the Pythagorean Theorem tells us that

$$|AP|^2 + |BP|^2 = |AB|^2$$

Now the left-hand side depends upon P but the right-hand side does not. That tells us that as P moves around the circle, the sum of the squares of its distance from B and A remains constant.

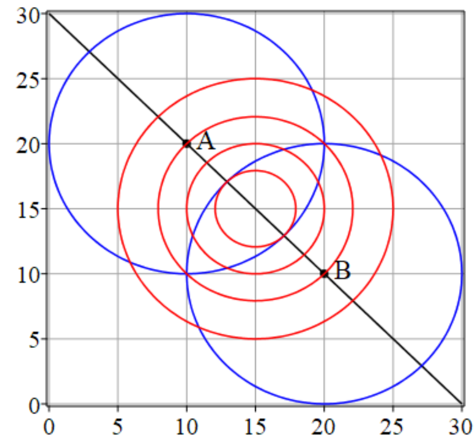


But look what that means in terms of the distance graph. |AP| is simply the "X-coordinate" of the distance axes and |BP| is the "Y-coordinate." And as P moves on the circle, the equation tells is that $X^2 + Y^2$ is constant and is in fact the square of the diameter of the circle. Thus the trace of the red circle on the distance graph is a circle centred at the origin whose radius is the diameter of the original circle—twice the size! Neat huh?

More circles

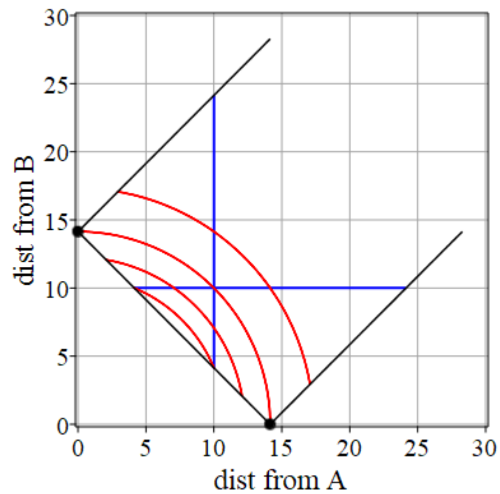
That piece of quarter-circle has me wondering. What if we let that middle circle change size? Do I still get a piece of a circle on the distance diagram?

And how does it "evolve" as the circle grows?



This is a good opportunity to make an animation and then sit back and watch it and be surprised! Over to Python!

Here's a few still shots from the animation. Does it make sense? Can you see why this is expected?

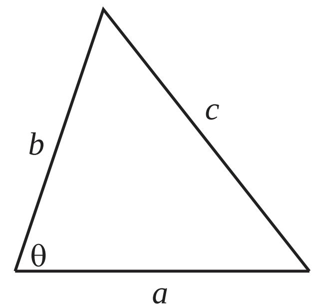


Those curves sure look like parts of a circle. Are they? Can you show that they are arcs of a circle?

And if they *are* circular, are they centred at the origin as before, and what is their radius?

The "generalized" Pythagorean Theorem.

For the special case in which the red circle passed through A and B, we used the Pythagorean Theorem to get the result. In the more general situation, what we need is an important generalization of the Pythagorean Theorem to the case of triangles that are not right-angled.

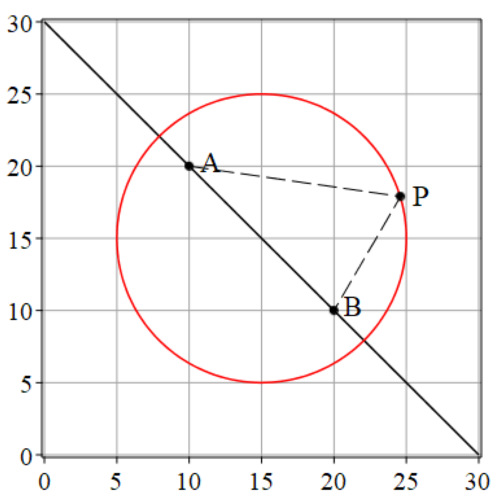


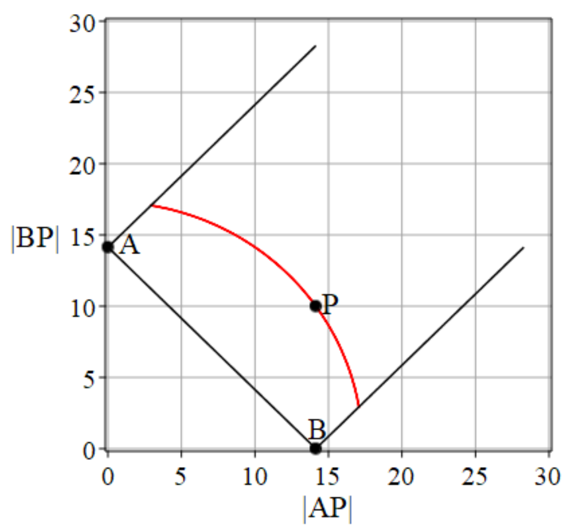
Consider the triangle at the right where θ is less than 90°. It's not hard to see that c^2 will now have to be *less* than $a^2 + b^2$. But how much less? Here's the general result:

$$c^2 = a^2 + b^2 - 2ab\cos(\theta)$$

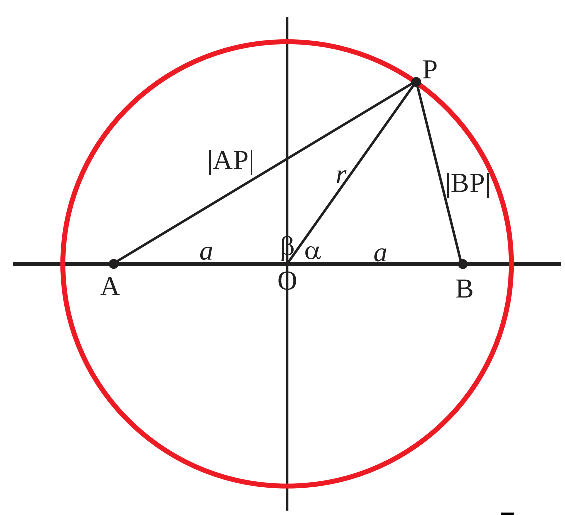
Note that if θ = 90, we recover Pythagoras. This result works for all angles θ, acute and obtuse.

Okay. On the left we have a circle of some radius r that does *not* pass through A and B, and on the right we have its image on the distance graph. The latter certainly does look like it might be a circle centred at the origin. To show that we need to show that for all P on the curve, $|\mathrm{AP}|^2 + |\mathrm{BP}|^2$ is a constant.





How do we do that? Apply the general formula above to the two triangles AOP and BOP:



$$|\mathrm{AP}|^2 = a^2 + r^2 - 2ar\cos(\alpha)$$

$$|\mathrm{BP}|^2 = a^2 + r^2 - 2ar\cos(\beta)$$

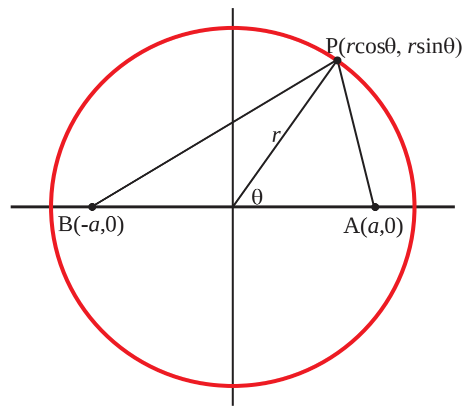
Okay. I want to add these equations, but what about the last terms? Well $\beta = 180 - \alpha$ and it follows that $\cos(\beta) = -\cos(\alpha)$, so when we add the equations, the last two term cancel:

$$|\mathrm{AP}|^2 + |\mathrm{BP}|^2 = 2(a^2 + r^2)$$

This tells us that on the distance graph we have a circle centred at the origin of radius $\sqrt{a^2 + r^2}$.

Analytic geometry proof.

In order to work more simply with the red circles in the landscape graph, I will give it a new coordinate system such that the AB line is the x-axis and the origin is halfway between A and B. Then the red circles are all centred at the origin. Take one of these and let it have radius r. We want to show that its image in the distance graph is a circle. In the diagram below, we have used $r = 10$, but the argument we use works for any r.



Here's the argument. Let the distance between A and B be $2a$ (where in this case $a = \sqrt{50}$). Then in the new coordinate system, A and B will have coordinates $A(a, 0)$ and $B(-a, 0)$. A general point P on the circle will have coordinates $P(r\cos\theta, r\sin\theta)$. Now work out the *square* of the distances between P and A. and P and B:

P to A: $${D_{PA}}^2 = \|(r\cos\theta, r\sin\theta) - (a, 0)\|^2 = (r\cos\theta - a)^2 + (r\sin\theta - 0)^2$$

B to P: $${D_{PB}}^2 = \|(r\cos\theta, r\sin\theta) - (-a, 0)\|^2 = (r\cos\theta + a)^2 + (r\sin\theta - 0)^2$$

Simplify:

P to A: $${D_{PA}}^2 = r^2\cos^2\theta - 2ar\cos\theta + a^2 + r^2\sin^2\theta = r^2 - 2ar\cos\theta + a^2$$

P to B: $${D_{PB}}^2 = r^2\cos^2\theta + 2ar\cos\theta + a^2 + r^2\sin^2\theta = r^2 + 2ar\cos\theta + a^2$$

These two expressions differ only in the sign of the middle term . If we add them, we get a simple expression:

$${D_{AP}}^2 + {D_{BP}}^2 = 2(r^2 + a^2)$$

Nice. Now what does that tell us? Well if for a moment I was to call D_{AP} x and D_{BP} y this would describe a circle about the origin of radius R where

$$R^2 = 2(r^2 + a^2)$$

In this case r =10 and $a = \sqrt{50}$, so that $R^2 = 300$, and

$$R = \sqrt{300} = 17.32$$

From the diagram, this looks about right.