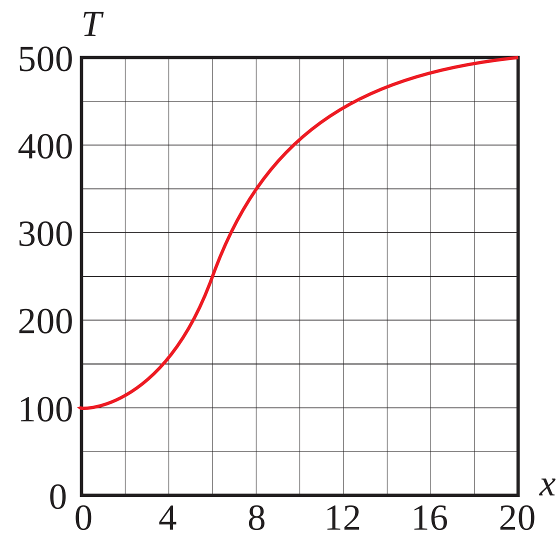
Advertising the Concert

We have a concert coming up at the school and we want to know how much to spend on internet advertising. The more we advertise, the more tickets we'll sell (up to a point), but also, the higher will be our advertising costs. What should we do to maximize our profit?

To answer this we need to know how expected ticket sales *T* depend on the number *x* of advertising spots we buy and this is plotted at the right. For example, if we buy 8 Kspots of advertising, we can expect to sell about 350 tickets. (A Kspot is a thousand spot ads delivered to a targeted set of viewers).



Ticket sales vs. advertising (Kspots)

Even with no advertising (x=0), we will sell 100 tickets, no doubt to family and friends of the performers, and supporters of the school. As *x* increases, *T* increases, slowly at first, then more quickly (as more folks spread the word), then slowly again as *T* gets above 400. This gives what's called an S-shaped curve: increasing slowly at the beginning and the end, but quickly in the middle.

Okay, we want to maximize our profit—that's the difference between revenue from tickets sales and advertising costs. So we need to know these.

Suppose tickets to the concert sell for $5 apiece and advertising costs $50 per Kspot. How much advertising should we buy to maximize our expected profit *P*, defined as the difference between the revenue *R* from ticket sales, and the total advertising cost *C*. Fill in the entries of the given table as accurately as is reasonable and use these to obtain an estimate of the value of *x* that maximize *P*. [The table gives us x = 14.]

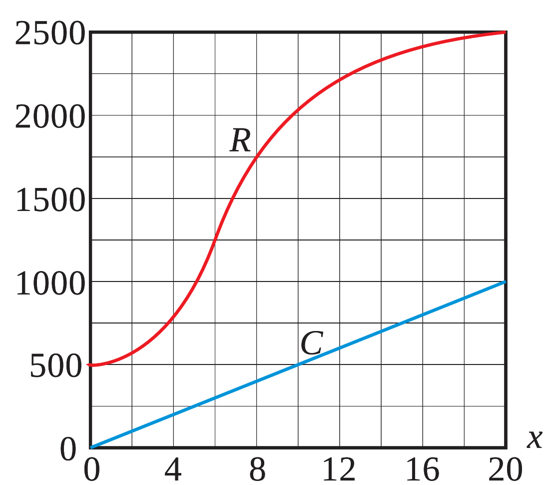
x	***T***	***R*=5*T***	***C*=50*x***	***P*=*R*-*C***
0	100	500	0	500
2	115	575	100	475
4	158	790	200	590
6	250	1250	300	950
8	350	1750	400	1350
10	408	2040	500	1540
12	442	2210	600	1610
14	470	2350	700	1650
16	482	2410	800	1610
18	492	2460	900	1560
20	500	2500	1000	1500

Our real objective here is to obtain a graphical solution, one that will apply to all values of *x*, not just the 11 values tabulated here. To do that we need to be able to "see" the profit *P* on the graph.

The simplest way to do that is to have a graph of revenue *R* and cost *C* against *x* on the same set of axes.

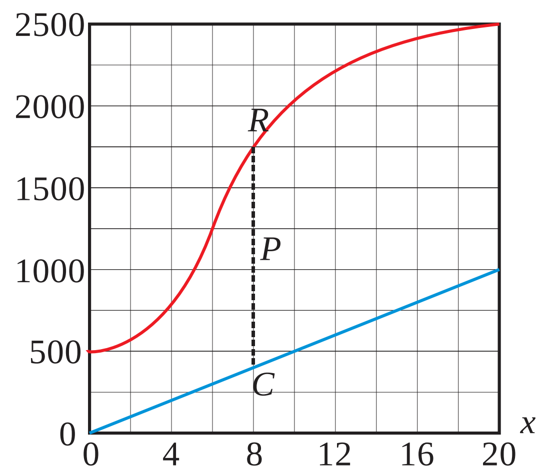
Now *C* is easy—it's just 50*x*, a straight line of slope 50.

But how do we get the *R*-graph? Well *R* is just 5*T* so we'll get a graph of *R* against *x* by simply multiplying the *T*-scale by 5. That's what we've done at the right.



Ticket revenue R and advertising cost C plotted against #Kspots x

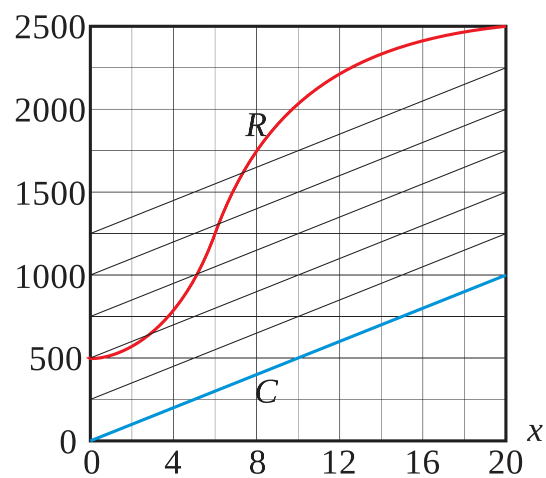
As you see from the workbook, at this point I pretty well leave the students to play with the configuration. It's a good small group exercise, with the possibility of sharing between groups. Groups can work with pencil and paper handouts or at the blackboard or on a white board with a rough copy of the graph. A smartboard will allow an exact copy.



Some students will need help simply to identify $P = R - C$ as the vertical distance between the two graphs. Given that, we are looking at the graphical problem of maximizing the length of a family of vertical line segments. How can we do that?

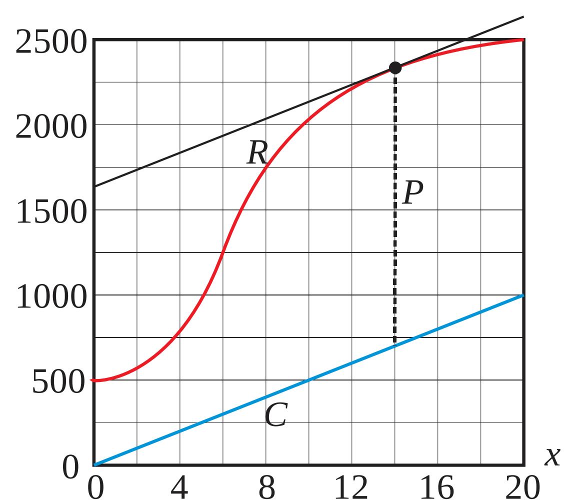
This is a good problem to support student presentations––"Here's how we looked at the problem." There are different ways to think about the problem.

A global argument
Here's one way. Take the C-line and start moving copies of it up vertically parallel to itself. The points on any particular translated copy are all the same distance from the C-line and thus it is easy to see which points on the R-curve are closer to the C-line and which are farther from it. For example, for points on the highest such line drawn at the right, the R points to the left of the intersection have lower profit and the R points to the right have higher profit.



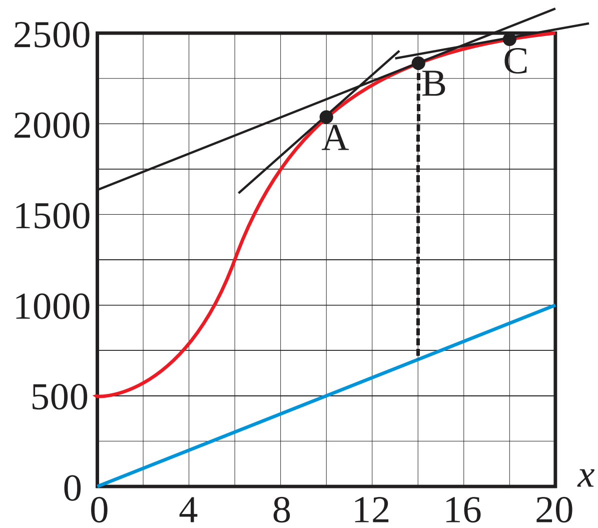
This makes it clear that we should keep moving the line up until it last touches the R-curve and that final point of tangency will then be the point of maximum profit.

Using this approach, the optimum investment in advertising appears to be very close to $x = 14$ Kslots giving us a profit of around $P = \$1650$. This is the same answer that we obtained from the table above.



This is what I would call a *global* argument. The tangent line which hits the R-curve at a single point leaves all other points on the curve below it. That provides a direct argument that all other points have a lower profit. Thus the point of tangency is the point of maximum profit among all possible values of x.

A local argument
I find a few students thinking they might compare the rate at which both revenue and cost increase at different points. This is a *local* argument in that they are wondering at any point whether they should increase or decrease their level of advertising.

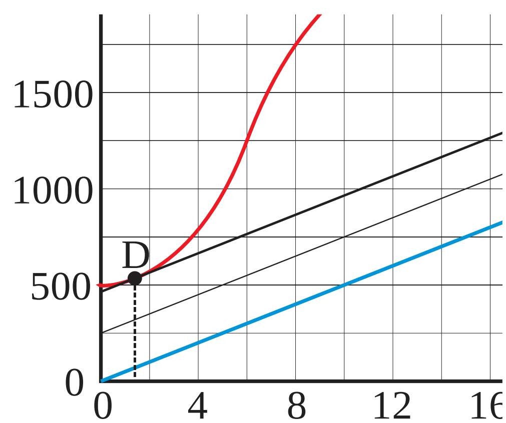


For example, point A corresponds to an investment of 10 Kspots of advertising, and if that's the level we are considering we can ask, should we invest more? Well if we invest more, both revenue and cost will increase, but the *R*-curve is steeper than the *C*-curve at this point so the revenue will increase more quickly than the cost. Thus, we should increase the advertising level *x*.

On the other hand, at point C (18 Kspots) the revenue is increasing more slowly than the cost. So instead we focus on the effect of *decreasing x* and we argue that our cost will go down faster than our revenue.

Using this kind of thinking, we converge to the optimum point B where *R* and *C* increase at the same rate with increasing *x*. We want $x = 14$ Kslots of advertising giving us a profit of around $P = \$1650$.

Some students might notice that there is another point D where *R* and *C* increase at the same rate and that's down near the bottom of the *R*-graph.



That's a great observation. B and D are both *equilibrium points* for the problem—in calculus language they are points at which $dP/dx = 0$ (since *R* and *C* change at the same rate). But B is a local *maximum* of the profit function (profit is lower on either side) whereas D is a local *minimum* (profit is higher on either side).