11. Arithmetic & Geometric Growth

(1) *The arithmetic sequence*

I start with \$20. Each day I am given \$3. How much do I have after 25 days?

Solution. Let's track it. We use n to keep track of the days and x_n to denote the amount I have after n days.

n	x_n
0	20
1	23
2	26
3	29
4	32
5	35
6	38
25	?

We tabulate our results at the right. After 25 days we have added \$3 a total of 25 times:

$$x_{25} = 20 + 25(3) = 95$$

We have \$95.

Formally we write the update rule as a recursive equation:

$$x_{n+1} = x_n + 3$$

This equation tells us how to go from one value of x to the next one: you add 3. But that's not enough information to determine the x_n. We need to get started:

$$x_{n+1} = x_n + 3 \qquad x_0 = 20$$

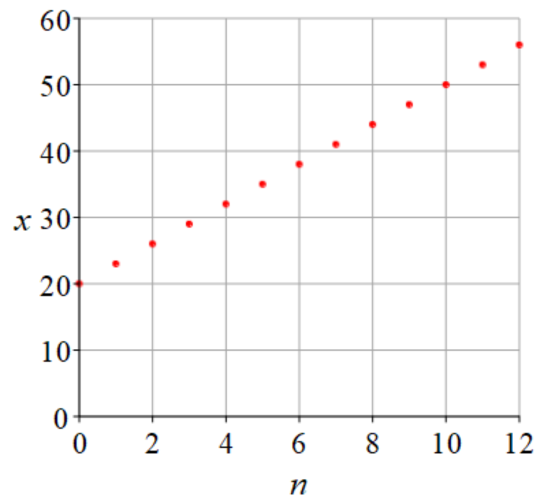
There. I have added an initial condition. These two equations completely determine the sequence.

The general formula for x_n that we get from this is

$$x_n = 20 + 3n$$

That's called the "solution" of the equations.

The sequence $\{x_n\}$ is graphed at the right. Of course, the points lie in a straight line!



Definition.
An *arithmetic sequence* is a sequence $\{x_n\}$with the property that each term is obtained from the preceding term by the addition of a *common difference* *d.* Formally, the sequence is specified with a recursive equation and an initial value:

$$x_{n+1} = x_n + d \qquad x_0 = A$$

We can solve this equation to get the general formula:

$$x_n = A + nd$$

Note: Here "arithmetic" is not a noun but is an adjective. As such it is pronounced "a-rith-meh'-tik"with the emphasis on the *meh*, as in the pronunciation of "geometric".

(2) ***The geometric sequence***

You put $100 into an account which grows by 12% every year. How much is in the account after *n* years?

Solution. Again we let x_n denote the amount I have after *n* years. What we want is a recursive equation, a formula for x_{n+1} in terms of x_n.

It's not immediately clear how to get that. How do you iterate a percentage increase?

Let's see. Suppose I have an amount *x*. How much will I have after one year? Well if it grows by 12%, it increases by an amount 0.12*x*. The new amount is then:

$$x + 0.12x = (1 + 0.12)x = 1.12x$$

That's a key step––factoring the *x* out to get a multiplier–– every year the amount is multiplied by 1.12. We know how to iterate that.

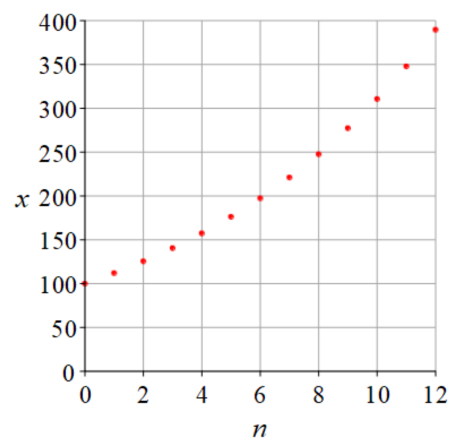
$$x_{n+1} = 1.12x_n \qquad x_0 = 100$$

I have added the initial condition and these two equations now determine the sequence. Each year we multiply by 1.12. In *n* years we multiply by 1.12 a total of *n* times:

$$x_n = 100(1.12)^n$$

The data are plotted at the right. They do not lie in straight line but form a concave-up curve. Why is that? The amount of the annual increase increases each year because the amount in the account is increasing.

n	x_n
0	100.00
1	112.00
2	125.44
3	140.49
4	157.35
5	176.23
6	197.38
7	221.07
8	247.60
9	277.31
10	310.58
11	347.85
12	389.60



Definition.
A *geometric sequence* is a sequence $\{x_n\}$with the property that each term is obtained from the preceding term by the multiplying by a fixed *multiplier*. Formally, the sequence is specified by giving a recursive equation and an initial value:

$$x_{n+1} = rx_n \qquad x_0 = A$$

We can solve this equation to get the general formula:

$$x_n = Ar^n$$

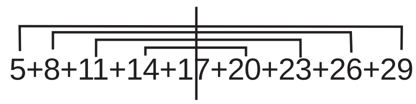
(3) *The sum of an arithmetic sequence*

Example 1. Calculate the sum:

$$S = 5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29$$

You could do this quickly on your calculator but if there were 100 terms instead of 9 that might not be such a great idea. So let's look for a pattern. The terms are easily seen to be an arithmetic sequence. We will exploit this to find a summation method which will work for any arithmetic sequence no matter how long.

There are many different ways to think about this but my choice is to group the terms in pairs symmetric about the "centre" of the sum:



Then

$$S = (5 + 29) + (8 + 26) + (11 + 23) + (14 + 20) + 17$$
$$= 34 + 34 + 34 + 34 + 17$$
$$= (4 \times 34) + 17$$
$$= 9 \times 17 = 153$$

What are the terms in the final sum--well 9 is the number of terms in the sum and 17 is the middle number. This will work with any arithmetic sum.

Example 2. Calculate the sum:

$$S = (-9) + (-2) + 5 + 12 + 19 + 26 + 33 + 40 + 47 + 54 + 61 + 68$$

This is an arithmetic sequence with 12 terms. What's the middle number? Well there isn't one and anyway it's a bit annoying to have to hunt the middle down. Much easier to say that the "middle of the sum" is half way between the first number and the last:

$$m = \frac{-9 + 68}{2} = \frac{59}{2} = 29.5$$

The sum is then

$$S = 12 \times 29.5 = 354$$

In fact, for an arithmetic sequence, this is the best way to think of the "middle" of the sum––as the *average* of the first and the last terms). We'll state that as a result:

This goes back to that famous story of the mathematician Gauss as a young lad in grade 6. It seems that the teacher, wanting to keep the class quiet for a hour, asked them to add up the numbers from 1 to 100. After less than a minute, Gauss came up with the answer $100 \cdot \frac{101}{2} = 5050$. The teachers was sure he had somehow cheated. But Gauss had simply thought about the structure of the summation.

> The sum of an arithmetic sequence $\{x_1, x_2, x_3, \dots, x_N\}$ is
>
> $$S = Nm$$
>
> where N is the number of terms and $m = \frac{x_1 + x_N}{2}$ is the average of the first and last terms.

(4) ***The sum of a geometric sequence***

Example 1. Calculate the sum:

$$S = 1 + 3 + 9 + 27 + 81 + \cdots + 3^{11} + 3^{12}$$

How do we do this? What we want to do is make use of the internal structure of the sum, the fact that each term is 3 times the previous term. Here's a nice trick: if we multiply all the terms by the common ratio 3, the new sum will still have lots of terms in common with the original sum. Let's see what we can do with that. Multiply both sides by 3:

$$3S = 3 + 9 + 27 + 81 + \cdots + 3^{12} + 3^{13}$$

These two expressions now have many common terms. Look what happen if we subtract them.

$$3S = 3 + 9 + 27 + 81 + 243 + \cdots + 3^{12} + 3^{13}$$

$$\underline{S = 1 + 3 + 9 + 27 + 81 + \cdots + 3^{11} + 3^{12}}$$

Subtract. Lots of lovely cancellations.

$$2S = -1 + 3^{13}$$

Remarkably enough that gives us a simple formula for the sum *S*:

$$S = \frac{3^{13} - 1}{2}.$$

If we follow the same argument for a general multiplier *r* we get:

> The sum of a geometric sequence with initial term 1 and multiplier *r* is
>
> $$S = 1 + r + r^2 + \cdots + r^{n-1} + r^n = \frac{r^{n+1} - 1}{r - 1}$$

Example 2. Without using the formula above, find the sum:

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2048}$$

Then use the formula to check your answer.

Solution. Calculate the partial sums, $1 + \frac{1}{2} = \frac{3}{2}$, $1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$ …
A pattern emerges:

$$S = \frac{2(2048) - 1}{2048} = \frac{4095}{2048}.$$

Finally we take the general case of a sequence that can start at any value *a*. In this case we can simply take the *a* out as a common factor.

> The sum of a geometric sequence with initial term *a* and multiplier *r* is
>
> $$S = a + ar + ar^2 + \cdots + ar^{n-1} + ar^n = a\frac{r^{n+1} - 1}{r - 1}$$

(5) ***Putting A & G together.***

This is a great small-group exploration. Students love finding patterns in numbers. The trick is to see how the pattern relates to the size of the three parameters, r, d and A.

The whole story about annuities, and even about investments, is about managing funds that change both multiplicatively (interest) and additively (deposits and withdrawals). The basic recursive equation that describes this process has the form

$$x_{n+1} = rx_n - d \qquad x_0 = A.$$

There are of course standard ways to "solve" this equation, but that can come in later years (though at the end I give a couple of simple approaches for the interested student). Here, I give the students some data generated by the equation, and see if they can put the pieces together from that.

Suppose you start with A dollars. Each day you double it and then take away $1. Your job is to figure out what happens after many days.

First formulate this as a recursive system. Let x_n be the amount after n days. We are told that:

$$x_{n+1} = 2x_n - 1 \qquad x_0 = A$$

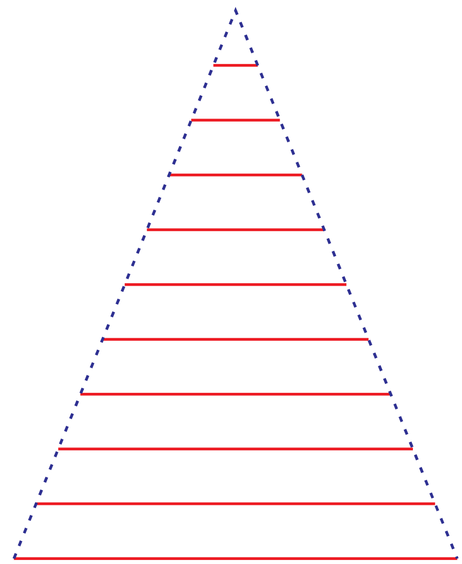
Of course what happens depends on the starting amount A. Thus you need to find a formula for x_n in terms of n and also, of course, in terms of A. As data to work with I give you the following table. It displays the terms of the sequence for different stating values A. What patterns can you see? Do they lead you to a general formula for x_n in terms of any A?

Values of x_n from the equation $x_{n+1} = 2x_n - 1$ $(x_0 = A)$							
n	A = 1	A = 2	A = 3	A = 4	A = 5	A = 6	A = 7
0	1	2	3	4	5	6	7
1	1	3	5	7	9	11	13
2	1	5	9	13	17	21	25
3	1	9	17	25	33	41	49
4	1	17	33	49	65	81	97
5	1	33	65	97	129	161	193
6	1	65	129	193	257	321	385
7	1	129	257	385	513	641	769
8	1	257	513	769	1025	1281	1537
9	1	513	1025	1537	2049	2561	3073
10	1	1025	2049	3073	4097	5121	6145
11	1	2049	4097	6145	8193	10241	12289
12	1	4097	8193	12289	16385	20481	24577

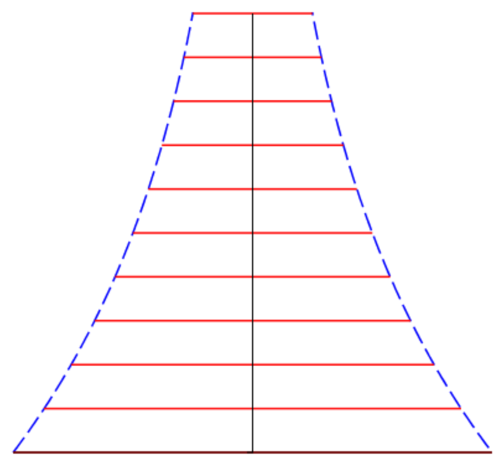
Answer: $x_n = (A-1)2^n + 1$

Problems

1. You put $100 into an account in which the balance is increased by ½% every month. How much is in the account after 24 months?
Answer: $100(1.005)^{24} = 112.72$



2. The diagram at the right shows a family of horizontal lines in a triangular pattern. The triangle has height 10 and base 8. In fact there are 100 lines in the pattern of which only 10 appear in the diagram, those at heights 0, 1, 2,   ,8 and 9. These 100 lines are equally spaced with distance 0.1 between neighbours. The bottom line (line 1) is at height zero and has length 8. The top line (line 100--not drawn) is at height 9.9.
(a) At what height is line 47 and how long is it?
(b) Find the sum of the lengths of all the 100 lines.



The diagram at the right shows a family of horizontal lines in a curved triangular pattern. The triangle has height 10 and base 8. In fact there are 101 lines in the pattern of which only 11 appear in the diagram, those at heights 0, 1, 2,   ,9 and 10. These 101 lines are equally spaced with distance 0.1 between neighbours. The bottom line (line 0) is at height zero and has length 8. The top line (line 100) is at height 10 and has length 2. The line lengths, from the bottom up, form a geometric sequence.
(a) At what height is line 47 and how long is it?
(b) Find the sum of the lengths of all the 101 lines.

Group Assignment
Suppose you start with A dollars. Each day you double it and then take away $5. Your job is to figure out what happens after many days.

First formulate this as a recursive system. Let x_n be the amount after n days.

$$x_{n+1} = 2x_n - 5 \qquad x_0 = A$$

You job is to find a formula for x_n in terms of n and the starting amount A. As data to work with you are given the following table of values.

Values of x_n from the equation $x_{n+1} = 2x_n - 5$ $(x_0 = A)$								
n	A = 5	A = 10	A = 15	A = 20	A = 25	A = 30	A = 35	A = 40
0	5	10	15	20	25	30	35	40
1	5	15	25	35	45	55	65	75
2	5	25	45	65	85	105	125	145
3	5	45	85	125	165	205	245	285
4	5	85	165	245	325	405	485	565
5	5	165	325	485	645	805	965	1125
6	5	325	645	965	1285	1605	1925	2245
7	5	645	1285	1925	2565	3205	3845	4485
8	5	1285	2565	3845	5125	6405	7685	8965
9	5	2565	5125	7685	10245	12805	15365	17925
10	5	5125	10245	15365	20485	25605	30725	35845
11	5	10245	20485	30725	40965	51205	61445	71685
12	5	20485	40965	61445	81925	102405	122885	143365

Answer: $x_n = (A - 5)2^n + 5$