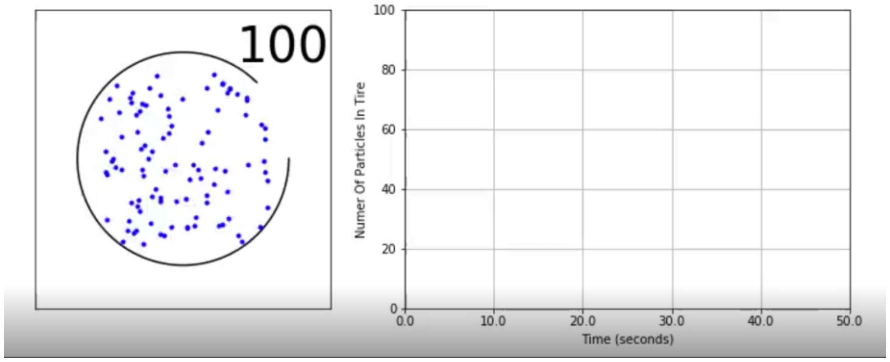
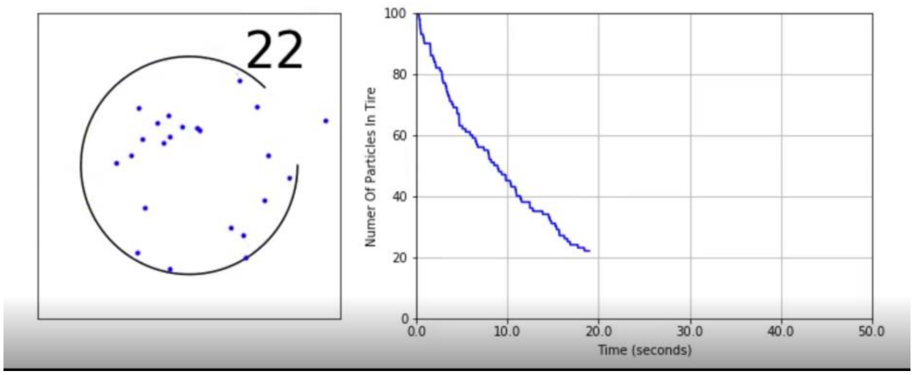
10. Exponential Growth and Decay.

The objective here is to give the students some kind of "hands-on" experience with this fundamental process od exponential change. In the past I have worked with large numbers of dice(for example) but I wanted something more visually striking. My RA Becca made a couple of python animation that I feel work really well. Over the past month I have worked with these with students and colleagues and the interactions I have had been very interesting. I'm impressed with the caliber of questions that arise as students watch these animations.



The tire animation captures the escape of molecules of air through the hole. We start with a 100 molecules with independent randomly positions and directions of movement, all moving at the same speed *v*. The students watch fascinated as the particles escape through the opening and they see the graph being plotted of the number remaining in the circle.

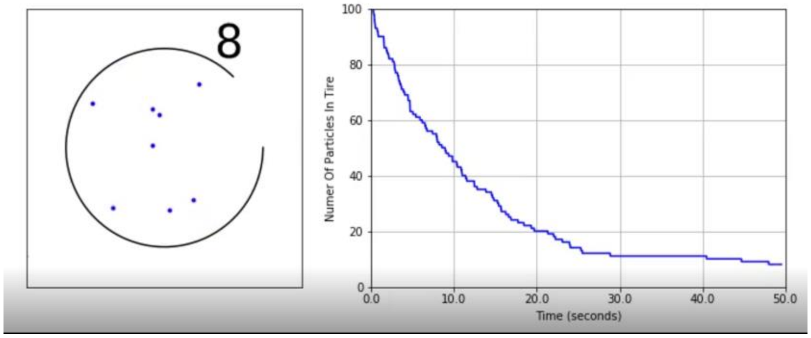


The first question, of course, is what kind of function is being plotted here. It is of course a random process so it won't exactly fit any simple mathematical form, but what is it supposed to be? What simple curve should the data fit?

The students respond that it should be exponential (and we have just talked about this in the tire model) but I press them––exactly what is it that allows you to draw this conclusion?

They are not so good at that. They say things like, well the more molecules that are in the circle, the faster they will go out. Is that enough to guarantee that the function is exponential? They are not so sure.

The precise condition of course is that the rate at which N decreases (the exit rate out of the hole) has to be proportional to N. Finally I get someone to say it in the right way.



I then ask for the mathematical form of the equation, and after a bit of discussion they settle on:

$$N = 100k^t$$

Where k is the 1-second multiplier. Of course the next question is: *what is the value of k*?

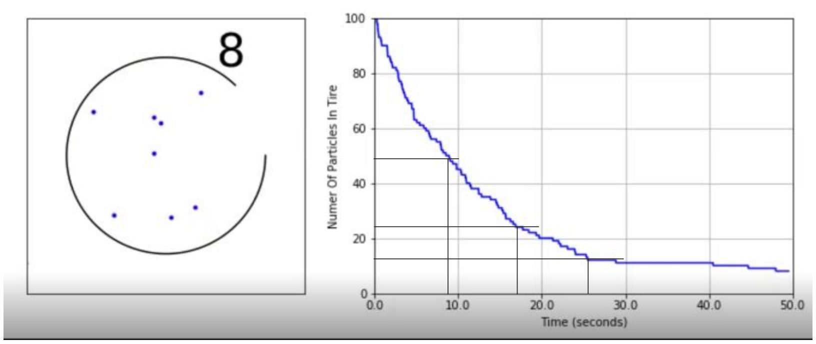
This is actually a lovely question and there are different ways to estimate and/or calculate k.

Estimating k from a data run.
One way to estimate k is to use the data to measure the half-life of the process, the time required for N to fall to half its current value. If the process is truly exponential, this should be same no matter where we start on the curve. Indeed let h be the half-life and take any fixed time t with $N = 100k^t$. Then at time $t + h$, the number of molecules should be half of N:

$$(0.5)N = 100k^{(t+h)}$$
$$= 100k^t k^h$$
$$(0.5)N = Nk^h$$
$$0.5 = k^h$$

and this is independent of N. If we know h we can solve this equation for k:

$$k = (0.5)^{1/h}$$



A half-life analysis of the graph provides evidence that the process is exponential, and gives us an estimate of h = 8.5 s for the half-life. That gives an estimate for k of

$$k = (0.5)^{1/h} = (0.5)^{1/8.5} = 0.92$$

Starting at N = 100, the time for N to drop to 50 looks to be 8 or 9 seconds, let's say 8.5 s. Then it should drop to 25 (half of 50) in another 8.5 s. Well that's about right––at t = 17 N is pretty close to 25. Adding another 8.5 s brings us to 25.5 s and N is certainly close to 12.5 at that time.

By this measure, the curve does seem exponential and h = 8.5 is a reasonable estimate of its half-life.

By the way if we add another 8.5 s, N has definitely *not* fallen to half its value. For small values of N, random effects play a major role and different runs will produce quite different results. However there is another reason to expect a departure from exponential behavior near the end of the run. We'll look at that in grade 12.

Exponential growth.
After Becca constructed the tire animation, we decided we ought to have a similar animation for exponential *growth*. But what to do? Certainly we want to close up the hole so that no-one escapes. And then we need a way to generate new molecules, a way such that the generation rate is proportional to N.

The first idea we had was that every time two molecules collided, they should produce a baby molecule (who would immediately become a new adult). That seemed like fun, but would it give us what we want? Would it produce exponential growth?

Well we need to decide how to define a "collision." Let's say that two molecules collide when their centres are within some specified distance δ of one another. Okay, given that, we want to know whether the growth rate of the population (which now equals the number of collisions per second) is proportional to N, the number of molecules in the circle.

Well that's a pretty interesting question –definitely one to throw out to the class as it will help to cement the key attribute of exponential growth.

It turns out that the answer is NO. It would be closer to the truth to say that the number of collisions was proportional to N^2. Here's the reason. If we follow a single molecule, the number of collisions it has per second should be proportional to N (or actually, to N–1). Now that will be the case for all N molecules, so that gives us $N(N-1)$ collisions. Actually that's wrong because we are counting every collision twice. So it's $N(N-1)/2$ collisions. That's per second. That's definitely not exponential—it's faster than exponential.

How can we fix that? Again, that's a good question for the class. We need to abandon sexual reproduction and move to cloning. What are the conditions under which a particle might be selected to reproduce? There are different ways to set that up and we tried a few of these.

The lightning model
What we need is for the number of cloning events per second to be proportional to N. Ultimately we decided on a "lightning storm" that struck periodically at a random place in the circle and every particle that was within a fixed distance δ of a strike would clone an offspring who would go off in a random direction and immediately become a new adult.

Is that exponential? Yes it is. The probability that a fixed particle will clone in any one-second interval is independent of the number of particles and is the same for all particles. It follows that the population-wide cloning rate will be proportional to population size. Growth rate is proportional to size.

This is a great discussion to have with in class with your students. What might we do with the tire animation to make it a model for exponential growth?

For example, the first question we ask here is a good one. It is essentially a model of sexual reproduction –-will it produce exponential growth?

I have asked my undergraduates and grad students this question. Many were unsure.

This is interesting. The idea that biological population, if unchecked, tend to grow exponentially may not be valid. It depends on how you model encounters.

Excellent question. Some design is needed here.

Suppose there is one lightning strike per second at a random point inside the circle and all individuals within distance $\delta = 0.7$ of the point of impact will clone.

We have set the parameters of the lightning model:
--One strike per second at a random point.
--particles within 0.7 cm of the strike will clone.

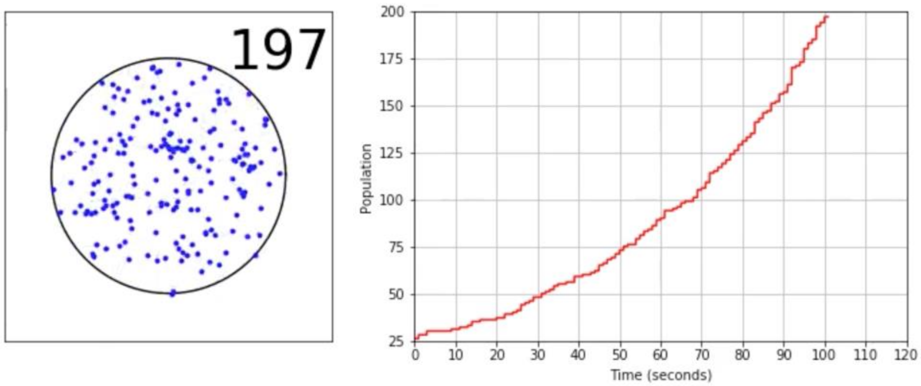
What will be the mathematical form of N in terms of time t? It will have the same form as the tire model. If we start with $N(0) = 25$ particles, then after time t the population size will be:

$$N = 25k^t$$

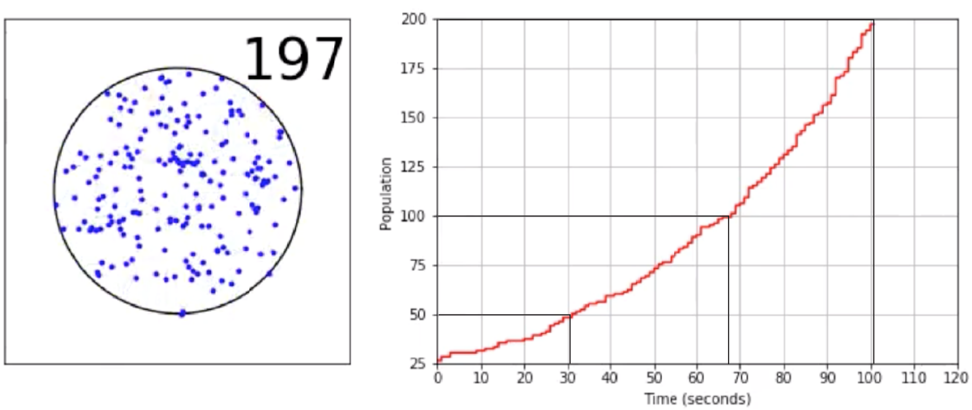
where k is the one-second multiplier. So how do we find k?

Well we are going to use two different approaches. First we build a simulation and use the data to provide an estimate of k. Then we "do the math" and obtain a theoretical prediction for k.

Estimating k from a data run.
Below is an example of a typical run of the lightning model.



In the tire model we used the data to estimate the half-life of the process. We can use the same approach here to estimate the "doubling time," the time required for N to rise to twice its current value. Again, if the process is truly exponential, this should be same no matter where we start on the curve. So let's start at 25.

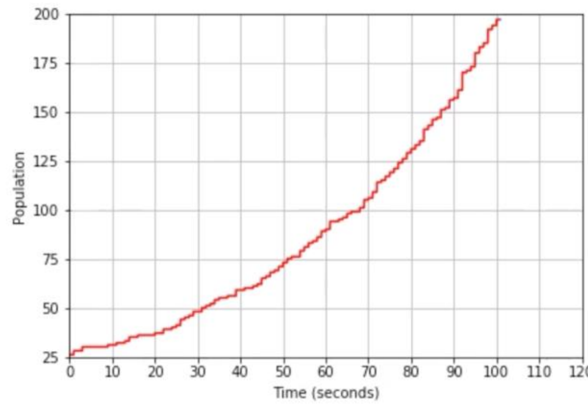


N	t	Δt
25	0	
50	30	30
100	67.5	37.5
200	100	32.5

Taking time readings at $N = 50$, 100 and 200, we get the estimates Δt of the doubling time of about 30, 37.5 and 32.5. These are close enough that they provide good evidence that the process really is exponential. The average of these values is $100/3 = 33.3$ and we take that as our empirical estimate of the doubling time.

The simulation data have given us an estimate of 33.3 seconds for the doubling time of the lightning model.

What does that give us for k?



Well the multiplier for any time interval of length t is k^t. If 33.3 is the doubling time, then

$$k^{33.3} = 2$$

and our estimate of k is:

$$k = 2^{1/33.3} = 1.19.$$

Calculating k from the parameters of the lightning model.
First note that k is the one-second multiplier for the population growth rate and thus if ε is the individual cloning rate (per second), then

$$k = 1 + \varepsilon.$$

So let's calculate ε. Fasten attention on a random particle. What is the probability that it will clone in any one-second period? Recall that there is one lightning strike per second at a random point inside the circle and all individuals within distance δ = 0.7 of that point will clone.

To simplify the calculation we will ignore boundary effects (strikes that are very close to the edge of the circle) and that allows us to assume that the circle of effect around a strike is contained inside the population circle.

Since the lightning strikes at a random point and the particles are also randomly located, the probability that a randomly chosen particle will clone at a particular lightning strike is the ratio of the area of the δ-circle to the population circle. Since the population circle has radius r = 5, this is

$$\frac{\pi\delta^2}{\pi r^2} = \frac{\pi(0.7)^2}{\pi(5)^2} = \frac{0.49}{25} \approx 0.02$$

This is our estimate of the cloning rate. On average, in any one-second period 2% of the population will clone. That gives us

$$k = 1 + \varepsilon = 1.02$$

and our lightning model theoretical equation is

$$N = 25k^t = 25(1.02)^t.$$

Let just check this with our empirical estimate of the doubling time (which was 33.3 seconds). The doubling time we get from the formula above is the solution t of the equation

$$(1.02)^t = 2$$

Using a calculator with "guess and check" gives us t = 35. That's certainly in close agreement.

We are solving the equation

$$(1.02)^t = 2$$

By "guess and check." In grade 12 you will learn how to solve it using logarithms

$$t = \frac{\log(2)}{\log(1.02)} = 35.0$$

Our theoretical calculations for the lightning model have given us a doubling time of 35 seconds. That agrees well with our empirical estimate of 33.3.