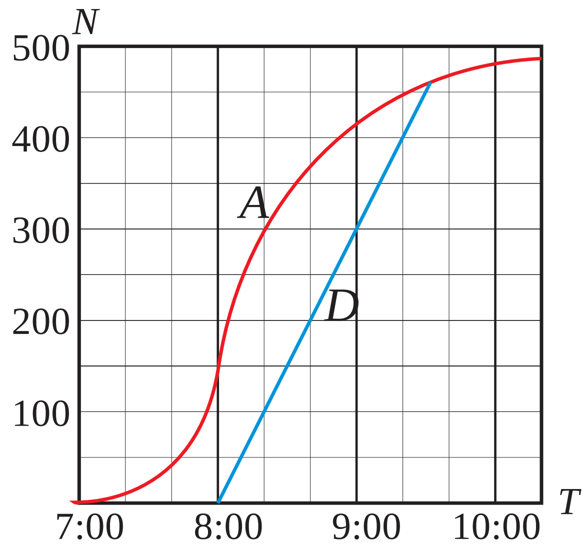
**Lineup**

*Problem 1.* The 500 slots for the annual Magnificent Mathematical Marathon (MMM) were available last year only on Pi-day. The registration desk opened at 8:00 that morning but already at 7:00 the lineup started to form. A couple of students doing a project on queuing theory monitored the arrival and departure times of the folks in line and their data are graphed at the right.

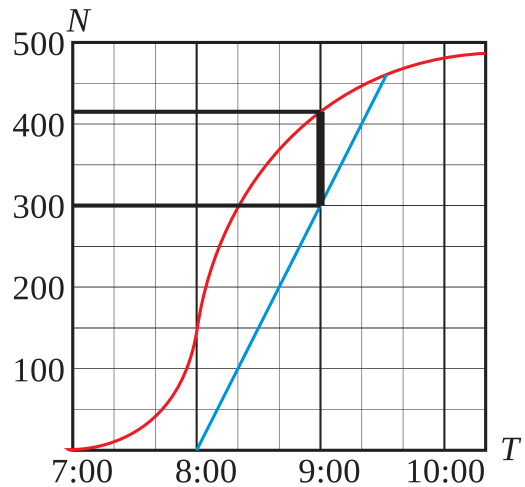



They found that folks were registered at pretty much a constant rate of 5 slots per minute (300 per hour). On the other hand, arrival times at the line-up varied greatly over the morning with a peak around 8 am of about 1 person joining the line every 4 seconds. By 10 o'clock most of the 500 slots had gone.

In each case, use the graph, perhaps with the help of a ruler construction, to give reasonable answers and diagram them on the graph provided. Note that we keep track of the people by naming them according to their time of arrival to join the line. Thus person 250 arrives at about 8:10 and is registered at about 8:50.

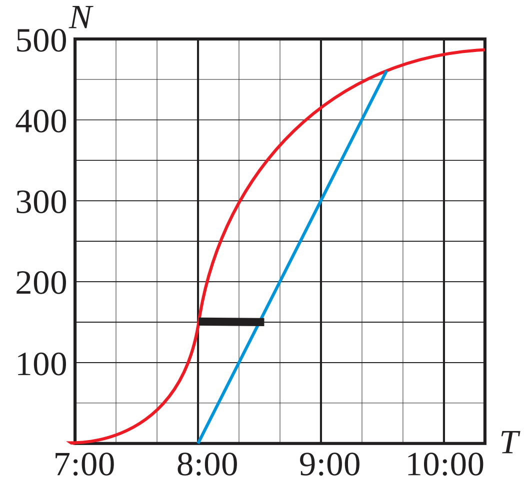
(a) At 9 am which person arrives, which person is being served, and how many people are in the lineup?

The answer is illustrated on the graph. At 9 am, person 415 arrives and person 300 is served. At this point the line is 115 in length. These numbers are all approximate.



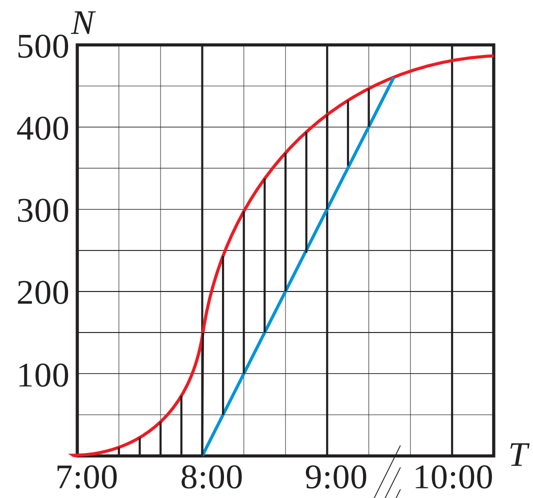

Some students will have trouble understanding this and it's important to come to grips with the following. The horizontal axis is time $T$ and the vertical axis is numbers of people $N$. Thus any statement about intervals of time will be represented by a *horizontal* line segment and any statement about numbers of people will be represented by a *vertical* line segment. For example, the length of the line at 9:00 is a vertical line segment because its units are numbers of people.

(b) How long does person #150 stand in line?



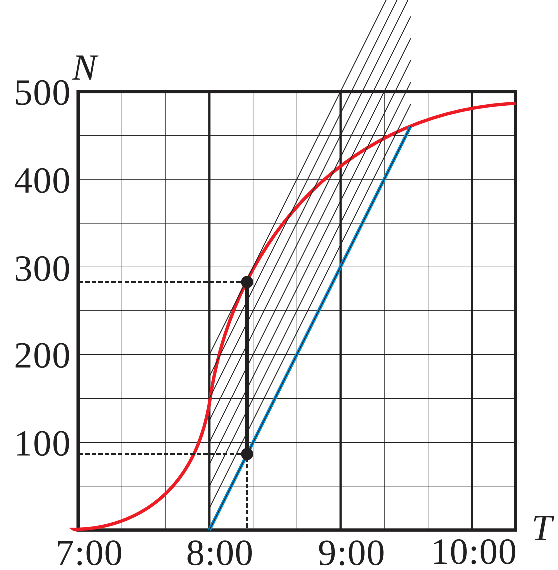

Person 150 arrives at 8 and is served at about 8:30; thus he or she stands in line for about 30 minutes. Note that this answer is illustrated with the length of a *horizontal* line as it gives us a time and the time axis is horizontal.

(c) At what time is the lineup the longest, and how long is it at that moment?




We are asked about the *number* of people in line so we will be thinking about the length of a *vertical* line segment. Now at any moment, the first person in line is the one being served and the last person in line is the one who has just arrived. That tells us that the folks in line run from the D-graph up to the A-graph. We can get a good picture of that family of line segments, but how do we pick out the one that is the longest?

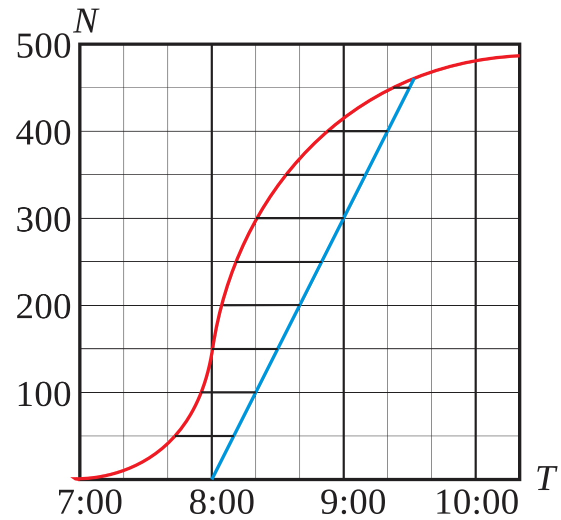
Using an idea from the advertising problem, we lift the D-line vertically parallel to itself until it last contacts the A-graph.




The last point of contact is at 8:18, the arrival time of person 282 (approx). At this time, person 87 (approx) is being served, giving us a lineup of length approximately 195.

We notice that this solution has ignored those folks who arrived between 7 and 8. For them, the length of the line is not the vertical distance from A down to D, but is rather the distance from A down to the *T*-axis. We can see directly that the line during this time is always less than 195; indeed its maximum is 150 attained at 8:00.

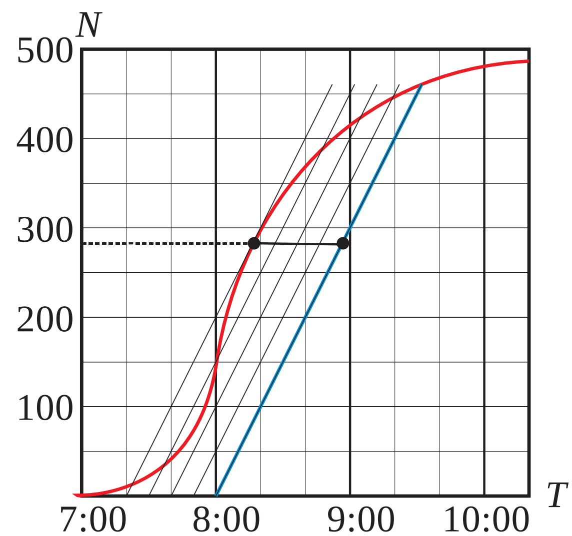
(d) Which person stands in line the longest and how long is that person in line?



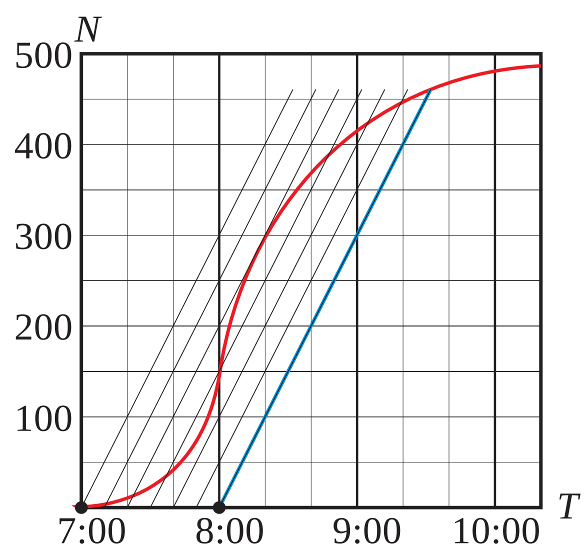

This time we are looking to maximize a *time* interval—the time, for any particular person, between arrival and departure. Thus we want the longest *horizontal* line between the A-graph and the D-graph.

So, as above, we translate the D-line parallel to itself but this time horizontally, until it last contacts the A-graph.

As we do this, the first thing we encounter is a tangency at the same point, #282, that we found above and we might think that this was the answer and that person 282 is the person who stands longest in line (40 minutes). And that might have been the case except for the early risers whose enthusiasm for the math marathon got them in line at 7 AM



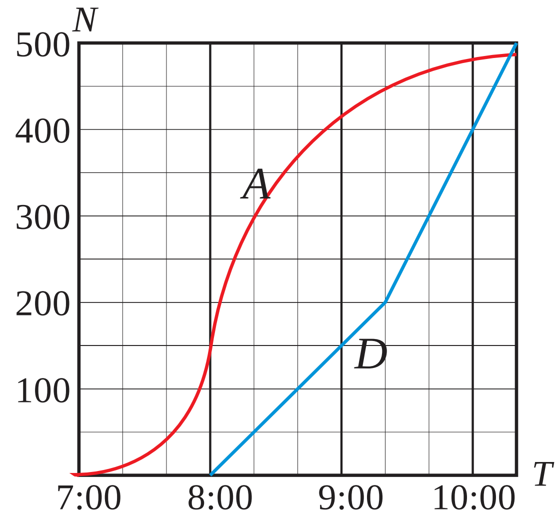

Indeed the last contact of the translated line with the A-graph happens right at the beginning with person #1 at 7 AM. He or she had a 60-minute wait for registration to open, but no doubt had a good math problem to work on.




The 40-minute lineup of person 282 is what is called a *local maximum* of the lineup time in the sense that the time in line was the longest among all those "nearby" individual who were standing in line just in front of or just behind person 285. But the *global maximum* belongs to person #1 who stands longer in line than all others.

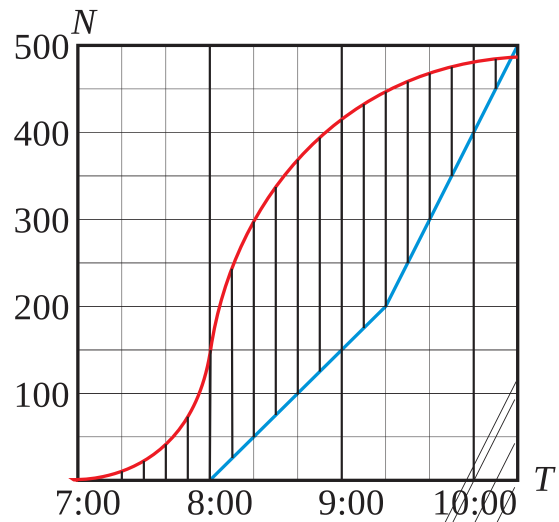
There is an interesting general question about the relationship between (c) and (d) above. Is the person who joins the lineup when it is longest always the one who waits the longest in line? If this is not always true, what *is* it true?

*Problem 2.* In Problem 1, when the registration desk opened, there were two servers each processing 2.5 tickets per minute. Now suppose that in fact only one server arrives at 8 with the second server running late and not arriving until 9:20, at which point the service rate increases to 5 persons per minute. Answer questions (c) and (d) as above under this new condition.



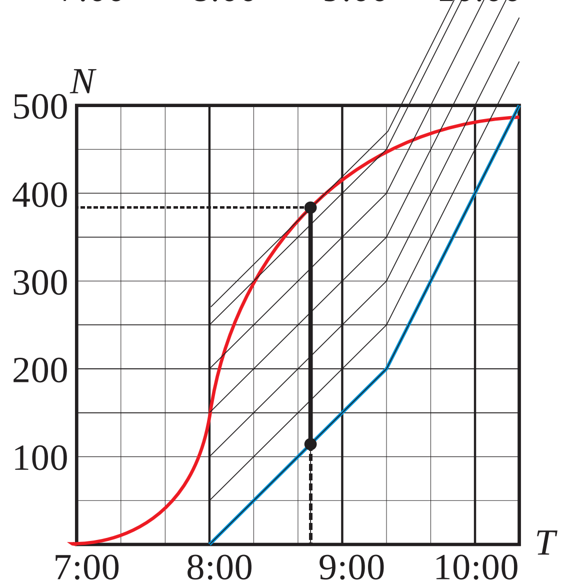

(c) At what time is the lineup the longest, and how long is it at that moment?

As before we are asked about the *number* of people in line so we will be thinking about the length of the *vertical* line segment from the D-graph up to the A-graph.

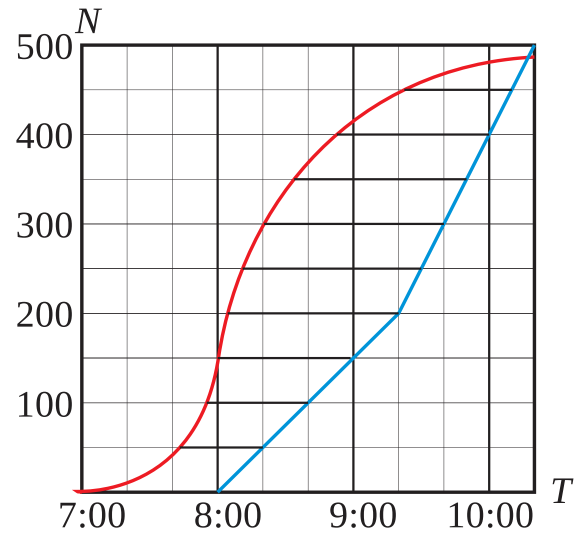



Again we lift the D-graph vertically parallel to itself until it last contacts the A-graph. We get a final point of tangency at about 8:46 when person 385 (approx.) is arriving and person 115 is being served, giving us a lineup of length approximately 270.

Again we have ignored those folks who arrived between 7 and 8, but the lineup during that hour never exceeds 150.

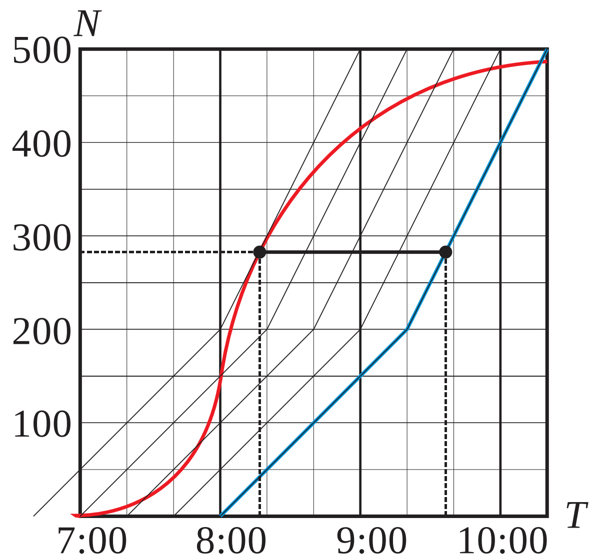


(d) Which person stands in line the longest and how long is that person in line?




This time we are looking to maximize the length of time a person is in line and this will be the *horizontal* line segment between the A-graph and the D-graph.

So, as above, we translate the D-graph parallel to itself but this time horizontally, until it last contacts the A-graph.




Surprisingly enough (or perhaps not surprisingly) we encounter the same point of tangency at person 282, that we found in Problem 1, except that in Problem 1 it was only a *local* maximum, but now it is the *global* max.

Thus person 282 is the one who stands longest in line and he or she is in line for 80 minutes.