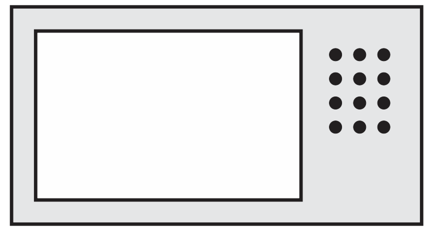
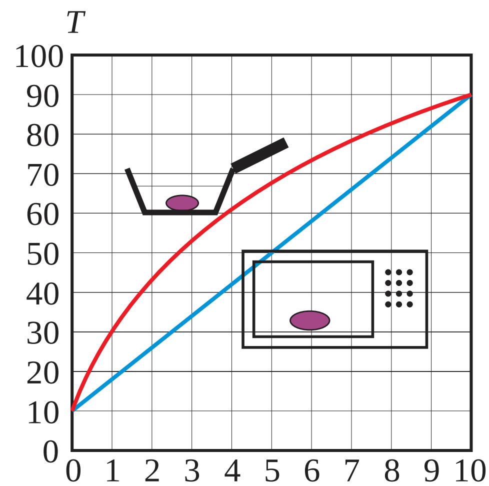
Raptor egg

I remember our first microwave oven. I would use it to heat up my mug of Ovaltine which I often had at night before bed. It had a probe which could be inserted into the mug and it would keep track of the temperature as it increased. And every time it went up one degree there was a tiny beep. And I recall noting particularly the regularity of the sequence of beeps––one could sing a song to it. That was my first clue that the temperature graph of the milk in my mug was a straight line.



We have a large raptor egg just out of the fridge (10°) which needs to be heated to 90°. Now there are two acceptable methods for cooking a raptor egg––one is to immerse it in boiling water (at a constant temperature of 100°), and the other is to microwave it at LOW power. By coincidence, it turns out that the cooking time is the same for both methods––exactly 10 minutes. The graph at the right provides the *T*-*t* (temperature-time) trajectories for each method over the interval $0 \le t \le 10$.



The difference between the graphs come from the different ways in which heat is transferred to the egg. In the microwave oven, heat is absorbed by the egg at a constant rate, and as a consequence, its temperature increases at a constant rate (of 8 degrees per minute).

However, in the boiling water, the rate of increase of *T* is high at the beginning and lower at the end, and seems to slowly decrease as the egg heats up. Indeed there is simple law governing this process. The rate at which the temperature of the egg increases is proportional to the *temperature difference* between the egg and the water. This makes sense––when the water is much hotter than the egg, heat energy is more quickly transferred to the egg.

Of course we can use either method but suppose we can also use a *combination* of the two––one method for a certain time and then the other, with a single switch. Can I lower my cooking time by switching? And if so by how much? To keep things simple, we assume the switch is instantaneous with no loss of temperature.

Newton's Law

The physical principle at work here is called *Newton's Law of Temperature Change*. It states that the rate at which the egg absorbs heat is proportional to the difference in temperature between the water and the egg.

Find a construction on the graph that solves the problem of how to shift between the two methods to minimize the total cooking time.

I have found this to be a good small-group problem. While most students manage to come up with the right answer, they offer a range of different kinds of arguments.

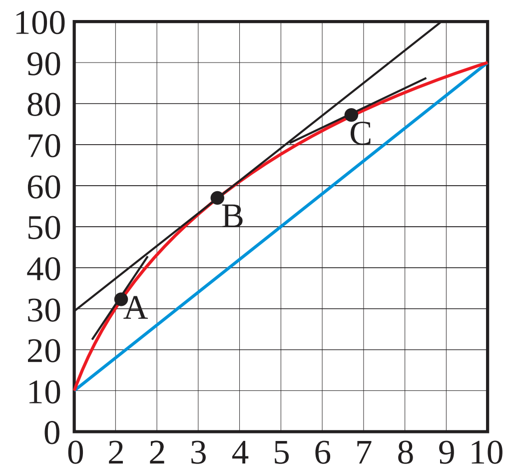
Again the various approaches my students come up with seem involve two different ways of thinking: *local* and *global*. The *local* approach argues that the minimum time is found by choosing, at each moment, the method that gives us the highest rate of temperature increase. The *global* approach works by constructing the entire trajectory and selecting the one with the smallest overall time.

The local argument. In its simplest form, the local argument asserts that we always want the rate of increase of temperature to be as big as possible. Now since that rate of increase is the slope of the graph, at each moment we choose the graph that has the biggest slope.

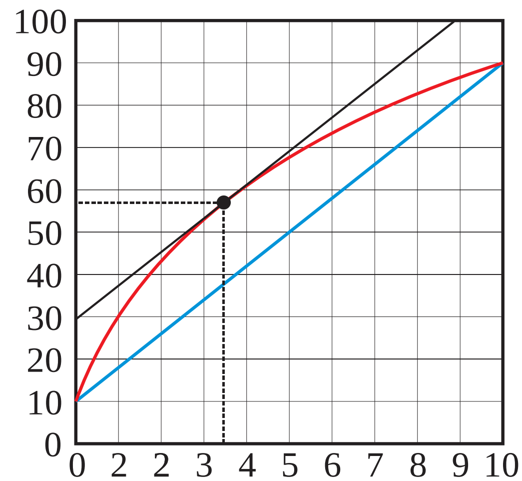
These are grade 9 students whose experience with rate of change and slope is pretty much confined to straight lines. Yet I find that they have little trouble extending this idea to non-linear relationships.

Notice that what makes this exceedingly simple argument work is our assumption that we could switch ovens at any time at no cost in terms of lost time or heat loss.

As we have noted, the slope of the boiling-water graph is higher at the beginning (when the egg is cool) than towards the end when the egg is closer to the water temperature. More precisely, at points like A the slope of the graph is higher than the slope of the microwave graph, whereas at points like C the slope of the graph is lower. Between A and C there is a point B at which the slope is exactly the same as that of the microwave graph. That would seem to be the temperature at which we should switch from the boiling water to the microwave. Using the graph we estimate this temperature to be close to 58°.

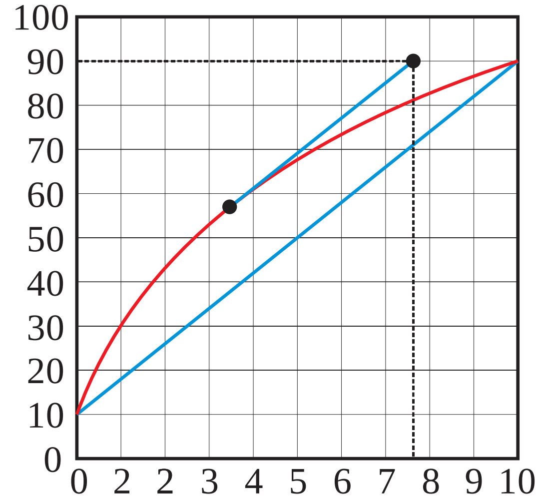


So that's the solution. Use the boiling water at the beginning and then when the slope of the temperature graph has become the same as the microwave rate (8°/min), switch to the microwave. This happens at an egg temperature of about 58°.



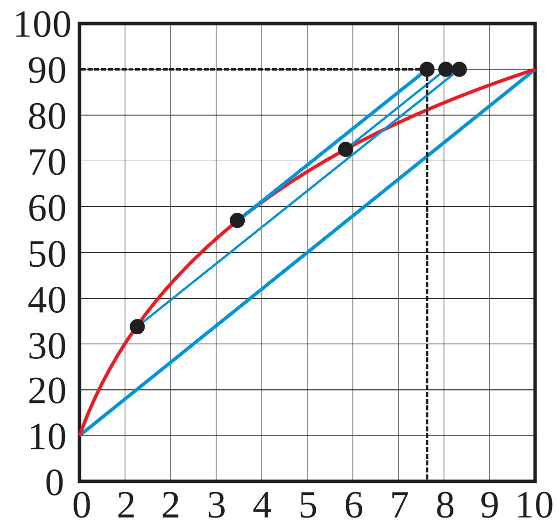
Note that I have reported the optimal switch point in terms of the *temperature* rather than the *time.* The reason is that it's the temperature of the egg that drives the action. The slope of the boiling-water graph is determined not by the time, but by the temperature of the egg.

When students give me this answer, I follow up by asking them for the resulting optimal cooking time. To find that, they pretty well have to draw the resulting overall temperature trajectory, and that is what gets them thinking in global terms.



Global thinking. The argument given above starts with the egg in the boiling water and switches when its rate of temperature increase falls to the microwave rate, 8°/min. The egg then remains in the microwave until it has attained the target temperature 90°. The trajectory is plotted at the right.

The total cooking time is then the time at which this trajectory hits 90°. This is seen to be close to 7.7 minutes. That's a saving of more than 2 minutes from each of the two "pure" strategies.

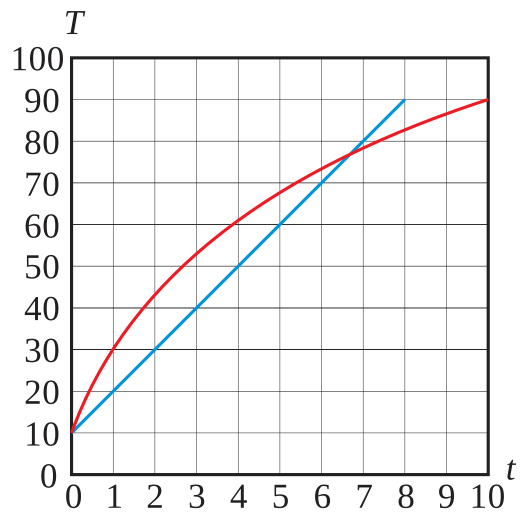


This construction can give us a way of proving that our switch point is the optimum, that there is no other switch point that would give us a shorter cooking time.

Simply take an arbitrary switch point, construct the trajectory and let the graph convince us that the cooking time will always be a bit longer. By way of illustration, I have made the construction at the right for two switch points, one above and one below the optimum. That's a global argument.

Problems

1. Recent advances in cooking have shown that raptor eggs can safely be heated in a microwave at a slightly higher setting, giving a temperature increase of 10°/min, thus requiring only an 8-minute cooking time. Supposing that the boiling water method is still available, follow the various approaches of the example to find the optimal point to switch from the boiling water to the microwave. Illustrate your steps on a copy of the graph, and estimate the total cooking time. Full explanations should be given.



2. It has been determined that the delicate flavour of a raptor egg will be bruised if the rate of change of temperature of the egg ever exceeds 12°/minute. Use a graphical approach to find the cooking program which minimizes the total time, but does not allow the rate of temperature increase to exceed 8°/minute. As usual you can assume that switches between methods happen instantaneously with no temperature loss.