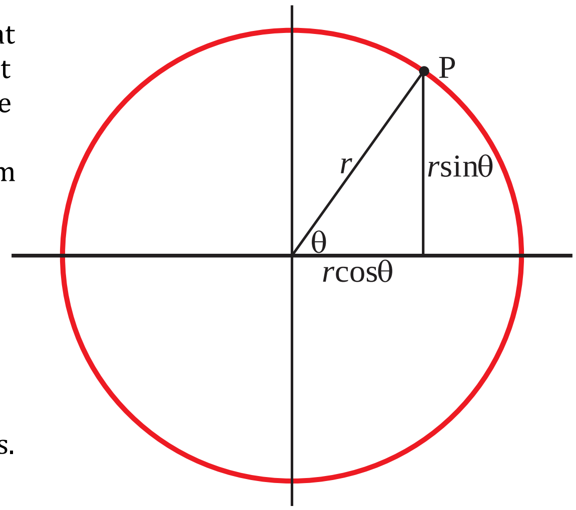
Circle and Spring

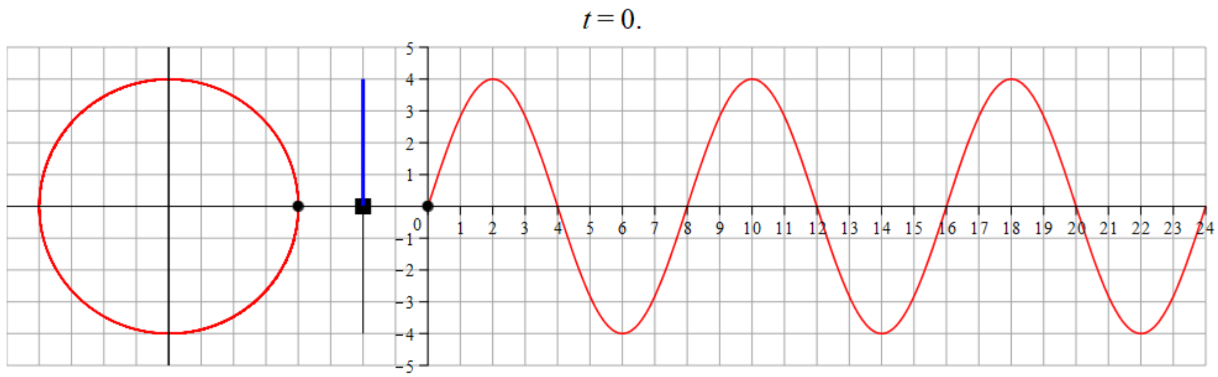
From time to time we will want to work with a point that is travelling around the circumference of a circle. In that case, we will need to describe its coordinates at any time *t*. If the circle is centred at the origin, a standard parameterization is to express the angle θ in the diagram as a function of *t*. In that case the point P will have coordinates



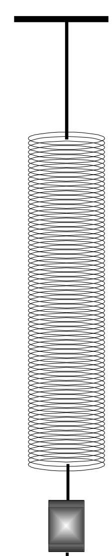
$$P(r\cos\theta, r\sin\theta)$$

where *r* is the radius of the circle.

To "see" the action we used an animation of the point P going around a circle of radius 4 at constant speed 45°/s. In that case the "height" of the point P (its *y*-coordinate) follows a sine curve, and the animation tracked that as well.



As the point moves counterclockwise around the circle, it follows the sign curve and we built the animation so that the small block in the middle goes up and down along with the point. The students agree that the oscillating block looks like it is on a spring. In fact one purpose of the animation is to make it plausible that the oscillation of the spring really is sinusoidal. That's actually not so easy to show without calculus and a bit of physics, and we will do that next year in grade 12. But here I want to use this fact to analyze some experiments with a spring. And the animation is a good way to start.

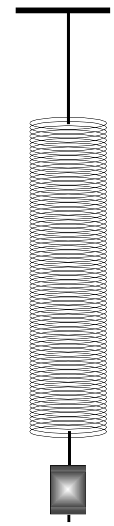


Indeed the sine-curve above has equation

$$y = 4\sin(45t) = 4\sin\left(\frac{360}{8}t\right)$$

where *y* is the height of the point P at time *t*. Here, the *amplitude* of the oscillation is 4 and its *period* is 8––that's the time for the block to complete one full oscillation (360°).

Working with the spring.
In the classroom, of course, we will need a hanging spring with a suitable weight on the end. Maybe there's one lying around, or one in the physics classroom, or maybe a strong elastic or a bungee cord will suffice. And it needs to be suspended from a fixed point––someone holding it up won't work. And we need to be able to measure things like period and amplitude.



The period.
First we need the period, the time for a complete oscillation. To reduce the error in the measurement it is best to take the time for, say, 10 oscillations and divide by 10.

But as soon as the students set up the spring system, they start asking questions. Does it matter how heavy is the mass? That is, does the period depend on the weight of the block? Does it matter how much we pull the weight down at the start? That is, does the period depend on the amplitude?

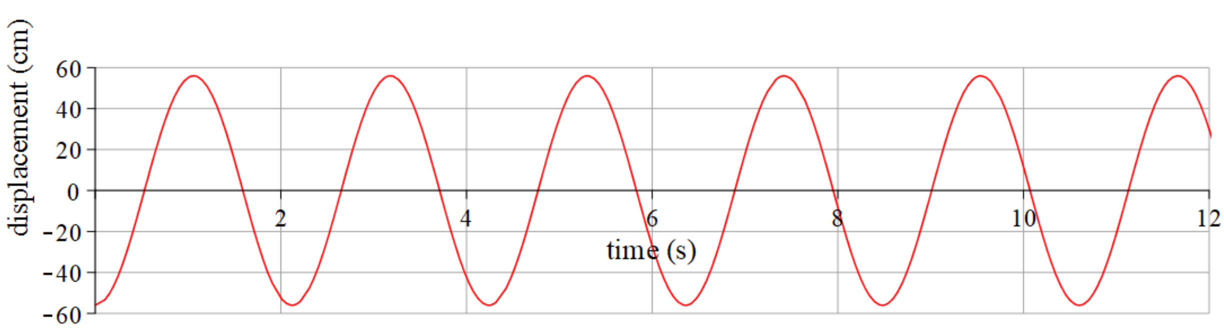
The answer to the first is yes––the period *does* depend on the weight of the block. Heavier blocks mean slower oscillations and longer periods. If we doubled the size of the block we'd multiply the period by $\sqrt{2}$––it's a square-root relationship. They will understand this interesting relationship better next year in Grade 12.

The answer to the second is no. The period is independent of the amplitude. The students can check this out by starting with a number of different displacements and then recording the time for 10 cycles. A table of data generated from the classroom is given at the right. That tells us that $P = 2.1$ s is a pretty good estimate of the period of our system and that's what we will use.

starting amplitude	*time for 10 cycles* (s)
60	21.2
50	21.1
40	21.2
30	20.8
20	21.0
10	20.8

To begin, the students pull the weight down 56 cm below its resting position. The sine curve they get has equation

$$y = 56\ \sin\left(\frac{360}{2.12}t - 90\right)$$



Damping.

So far we have ignored the fact that as the spring oscillates it loses energy to friction and the amplitude decreases. But how does it decrease? What sort of function is $A(t)$ the dependence of the amplitude on time.

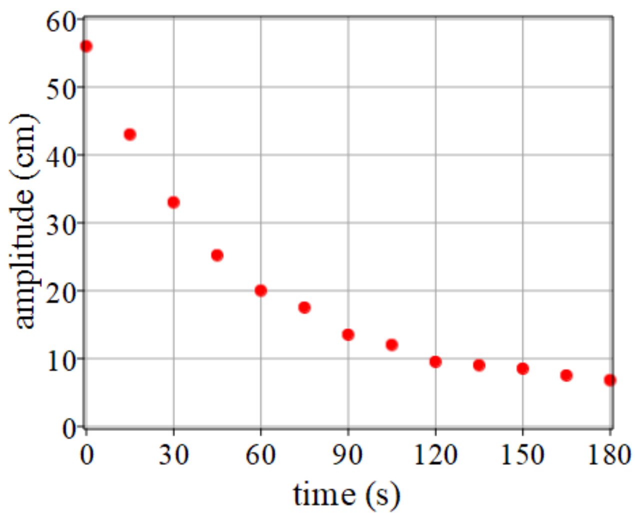
Time t (s)	Amp A (cm)
0	56.0
15	43.0
30	33.0
45	25.2
60	20.0
75	17.5
90	13.5
105	12.0
120	9.5
135	9.0
150	8.5
165	7.5
180	6.8

Well it turns out that the amplitude is a measure of the amount of energy in the system and as it oscillates, it loses that energy due to frictional forces. Now it is quite often true in such situations that *the rate of energy loss* in a system is proportional to the *amount of energy*. When that is the case the amount of energy in the system will decay *exponentially* and the graph of energy against time will be exponential

$$E(t) = E(0)r^t$$

where r is the multiplier belonging to one time unit. Let's see if that is the case for our spring system.

The table above provides data taken from a spring system in class over a 3 minute period. The amplitude A is measured in cm and the students took a reading every 15 seconds. The data are plotted at the right.



Are the data exponential?

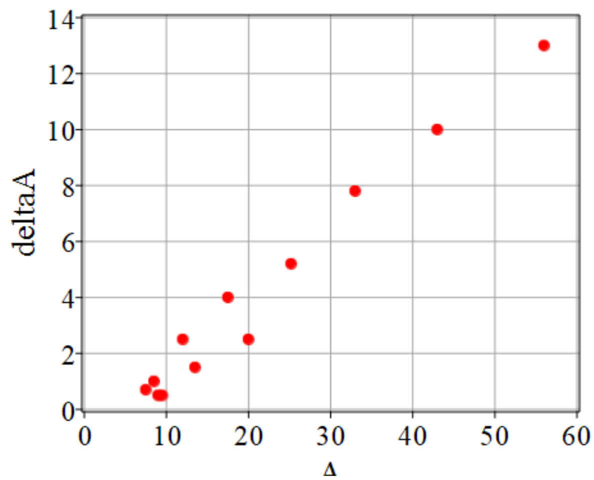
At the beginning (large amplitude) the data do seem to form a smooth concave-down curve that might well be exponential. As the amplitude decreases, the pattern is a bit more scattered. This might well be the effect of the reading error which is more pronounced for smaller values. As I recall the data were read by a student recording the minimum height of the weight with a vertical tape measure attached to the wall behind. The reading error here was likely at least 0.5 cm and that has a greater percentage effect on small values than on larger ones.

How does the rate of change of A depend on A?

Amp A	delta$A$
56.0	13.0
43.0	10.0
33.0	7.8
25.2	5.2
20.0	2.5
17.5	4.0
13.5	1.5
12.0	2.5
9.5	0.5
9.0	0.5
8.5	1.0
7.5	0.7
6.8	

Let's be more precise. If the amplitude did decay exponentially the change in A over each time interval would be proportional to A. That is, a graph of the change ΔA against A should be a straight line passing through the origin. Let's look at this.

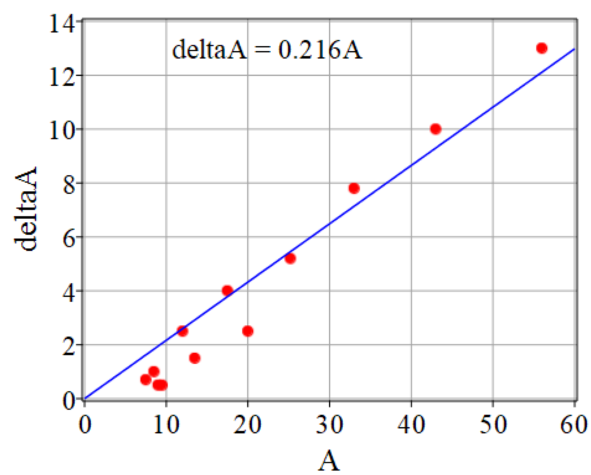
The table at the right calculates the change in A over each 15 s time interval against the value of A at the start of the interval and these data are plotted below. They do seem to lie in a fairly good straight line, though there is considerable scatter in the small amplitude range. We already noted above that we might expect that.



To get a better sense of the exponential character we calculate a least-squares fit to the data using a line passing through the origin. The line we get is

$$\Delta A = 0.216A$$

and this is plotted below.



A closer look at the graph suggests that the data might follow a different law for large amplitude than for smaller ones. The last three points (large amplitude) do lie along a line that passes through the origin with a slightly steeper slope, whereas the slope belonging to the small amplitude points might be lower. It is not uncommon for friction to act differently at low speeds than at high speeds.

The amplitude function A(t)
We are looking for an amplitude function of the form

$$A(t) = A(0)r^t$$

where r is the one-second amplitude multiplier.

Now we know what happens in 15 seconds––A is decreased by a factor of 0.216. For example

$$A(15) = A(0) - 0.216A(0)$$
$$= A(0)(1 - 0.216) = A(0)(0.784)$$

This is a good problem for the students. Given that A is decreased by a factor of 0.216 every 15 seconds, what is the one-second multiplier for the amplitude?

Thus 0.784 is the amplitude multiplier for a 15 second interval. What does that tell us about the one-second multiplier r? Well if r is the one-second multiplier, the 15-second multiplier will be r^{15} and hence:

$$r^{15} = 0.784.$$

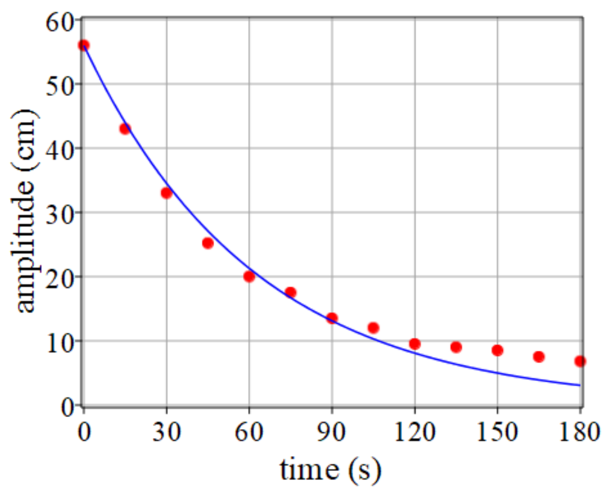
The one-second multiplier is then

$$r = (0.784)^{1/15} = 0.984$$

Thus our model for the amplitude at time t is

$$A(t) = A(0)(0.984)^t = 56(0.984)^t$$

and this is plotted at the right on top of our amplitude data.



The damped oscillation
Now it is time to put the periodic oscillation of the spring and the amplitude together to get a graph of the height of the weight over time. In fact, to avoid the oscillation from becoming a red blur, I have slowed the oscillation down by a factor of 5, using a period of 10.6 instead of 2.12.